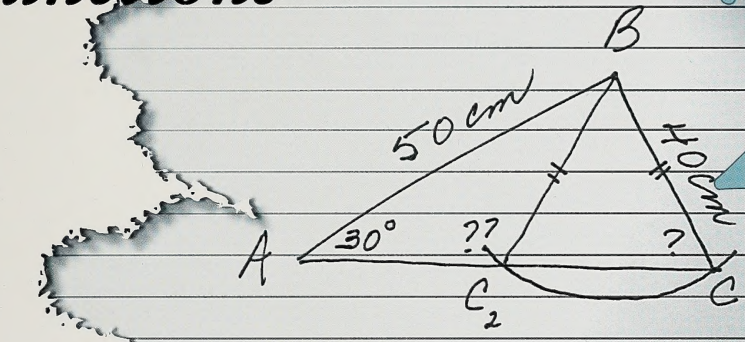
AL 2.3002-171
no ILL
v.4
University of Alberta Library
0 1620 3409745 9

# Module 4

# MATHEMATICS 20

## Functions




$$\sin C = \frac{c \sin A}{a}$$

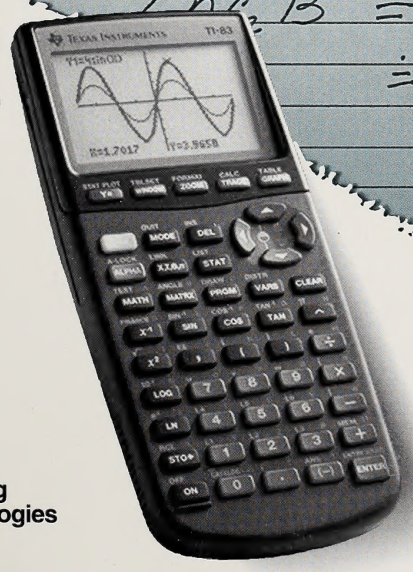
$$= \frac{50 \sin 30}{40}$$

$$\angle C \doteq 38.7°$$

$$\therefore \angle C_1 \doteq 38.7°$$

$$\angle AC_1B = 180 - \angle C_1$$

$$\doteq 141.3°$$




Learning Technologies Branch

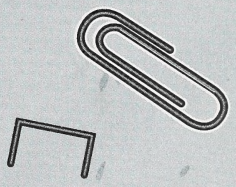
Alberta LEARNING

Digitized by the Internet Archive
in 2016 with funding from
University of Alberta Libraries

https://archive.org/details/puremathematics204albe_1

# *Pure* MATHEMATICS 20

## Module 4
## Functions



Learning Technologies Branch

Alberta LEARNING

Pure Mathematics 20
Module 4: Functions
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-1913-6

| This document is intended for | |
|---|---|
| Students | ✓ |
| Teachers | ✓ |
| Administrators | |
| Parents | |
| General Public | |
| Other | |

The Learning Technologies Branch has an Internet site that you may find useful. The address is as follows:

**http://www.learning.gov.ab.ca/ltb**

The use of the Internet is optional. Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

**ALL RIGHTS RESERVED**

Copyright © 1999, 2001, the Crown in Right of Alberta, as represented by the Minister of Learning, Alberta Learning, 11160 Jasper Avenue, Edmonton, Alberta T5K 0L2. All rights reserved. Additional copies may be obtained from the Learning Resources Centre.

No part of this courseware may be reproduced in any form, including photocopying (unless otherwise indicated), without the written permission of Alberta Learning.

Every effort has been made both to provide proper acknowledgement of the original source and to comply with copyright law. If cases are identified where this effort has been unsuccessful, please notify Alberta Learning so that appropriate corrective action can be taken.

**IT IS STRICTLY PROHIBITED TO COPY ANY PART OF THESE MATERIALS UNDER THE TERMS OF A LICENCE FROM A COLLECTIVE OR A LICENSING BODY.**

UNIVERSITY LIBRARY
UNIVERSITY OF ALBERTA

# Welcome

Welcome to Module 4. We hope you'll enjoy your study of Functions.

## Pure Mathematics 20

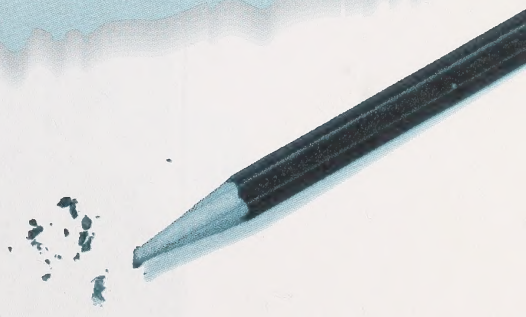
**Module 1**
Consumer Mathematics

**Module 2**
Equations and Inequalities

**Module 3**
Quadratics

**Module 4**
Functions

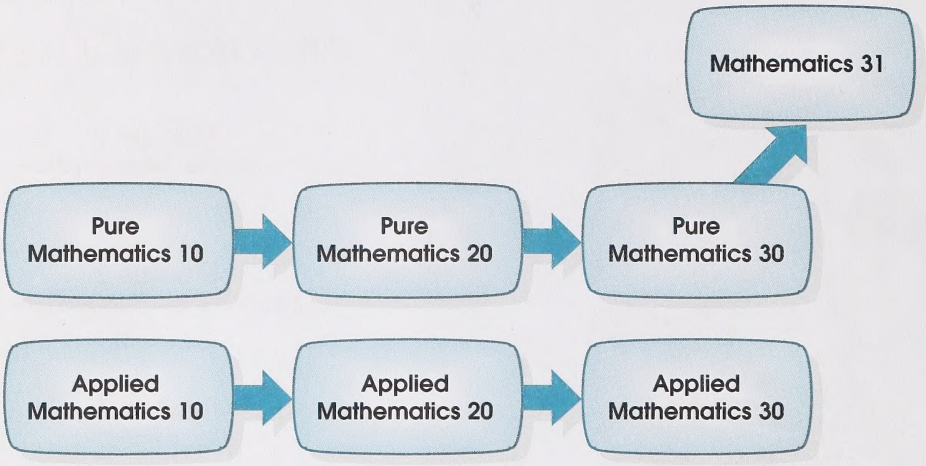
**Module 5**
Mathematical Reasoning and Geometry



Pure Mathematics 20 contains five modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.

# High School Mathematics Courses

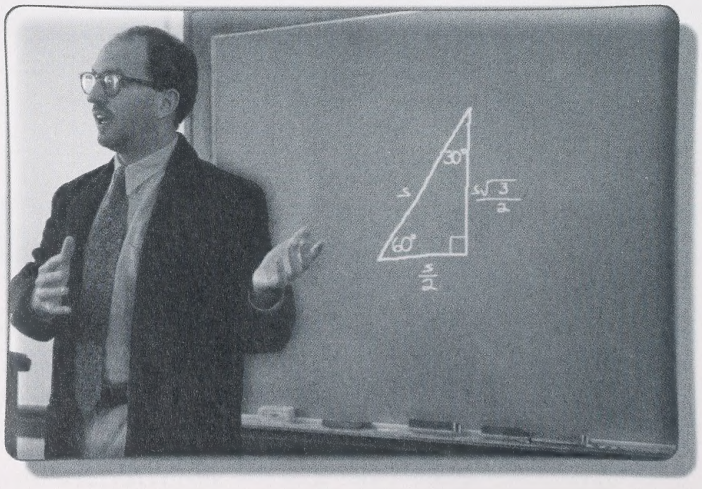
Pure Mathematics 20 is the second course in the Pure Mathematics 10–20–30 sequence of courses. Many students who take the Pure Mathematics 10–20–30 will also choose to take Mathematics 31. Another sequence of courses is Applied Mathematics 10–20–30.




## How Do the Sequences Differ?

Each sequence of courses is designed for students with different mathematical strengths and interests. The Pure Mathematics 10–20–30 sequence is intended for students who are strong in mathematical theory and algebra. The Applied Mathematics 10–20–30 sequence is better suited to students who prefer to solve mathematical problems using numerical reasoning or geometry.

Each sequence of courses is designed for students with different career plans. Students who plan to take courses such as commerce, engineering, physics, and engineering technologies in post-secondary education will need to take Pure Mathematics 30 in high school. Some of these students will also need to take Mathematics 31.

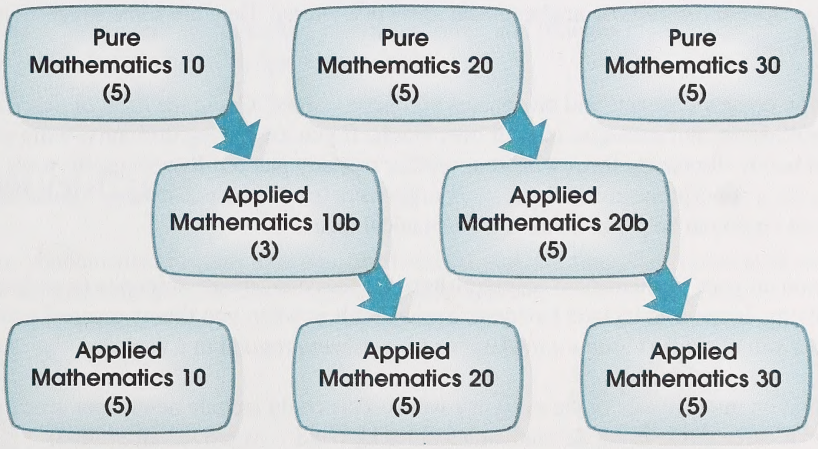


## Can I Transfer to the Applied Mathematics Sequence of Courses?

You should be aware that the applied and pure mathematics courses do have some topics in common; other topics are specific to either the applied or pure mathematics courses.

| Applied Topics | Common Topics | Pure Topics |
|---|---|---|
| • linear programming<br>• data tables and trends<br>• design and layout<br>• metric and imperial measure<br>• data presentation<br>• vectors and matrices<br>• periodic, fractal, and recursive patterns<br>• financial decision making<br>• costing and design problems | • spreadsheets<br>• line segments and linear graphs<br>• scaling<br>• triangles<br>• surveys<br>• financial mathematics<br>• quadratic functions<br>• circle geometry<br>• the bell curve | • irrational numbers<br>• exponents<br>• polynomial and rational expressions<br>• mathematical expectation<br>• growth patterns<br>• linear and non-linear systems<br>• operations on functions<br>• mathematical reasoning<br>• exponential and logarithmic functions<br>• conics<br>• combinations<br>• trigonometric functions |

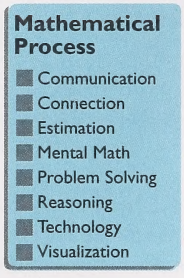
If you want to transfer from the Pure Mathematics 10–20–30 sequence to the Applied Mathematics 10–20–30 sequence at a future time, you won't have to repeat the topics that are common to pure mathematics and applied mathematics. If you decide to transfer to the Applied Mathematics sequence of courses after successfully completing Pure Mathematics 10, you may take the 3-credit course called Applied Mathematics 10b or you may take the 5-credit course called Applied Mathematics 10. If you decide to transfer after successfully completing Pure Mathematics 20, you may take the 5-credit course called Applied Mathematics 20b. The two bridging courses are shown in the following diagram:


Pure Mathematics 10 (5)
Pure Mathematics 20 (5)
Pure Mathematics 30 (5)
Applied Mathematics 10b (3)
Applied Mathematics 20b (5)
Applied Mathematics 10 (5)
Applied Mathematics 20 (5)
Applied Mathematics 30 (5)

# Mathematical Processes

Throughout the course you will be expected to do the following:

- communicate mathematically
- connect mathematical ideas to everyday experiences and to concepts in other disciplines
- develop and use estimation strategies
- develop and use mental math strategies
- develop and use problem-solving strategies
- reason and justify your thinking
- select and use appropriate technologies as tools to solve problems
- use visualization to assist in processing information, making connections, and solving problems




This icon shows which mathematical process you are expected to perform.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Pure Mathematics 20 to be your study partner. You will find that having a friend to discuss mathematics with will make your studying more enjoyable.

You are also encouraged to record your mathematical ideas in a journal. Here are some suggestions for organizing your journal:

- Keep a section on new concepts and procedures you have learned. Get in the habit of describing new concepts, procedures, and strategies in your own words. If you are having difficulty, write your questions in your journal before discussing them with your teacher or study partner. Record useful ways to help you remember what a concept means. Make graphic organizers (such as context webs, Venn diagrams, or organizational charts) to help you connect mathematical ideas.

- Keep a section on your mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you finally grasped a difficult concept (an "aha!") or when you first used a calculator or spreadsheet program in a new way.

- Keep a section on mathematics in the everyday world. This could include newspaper articles, cartoons, and descriptions of careers that require a mathematical background.

# Resources You Will Need

In order to complete Pure Mathematics 20 successfully, you will need to have access to the following resources, which can be purchased through the Learning Resources Distributing Centre:

- the *MATHPOWER™ 11* textbook, Western Edition, published by McGraw-Hill Ryerson (1999)
- a graphing calculator

  Any of the following graphing calculators may be used: *Texas Instruments* TI-83, TI-83 Plus, TI-82, TI-86, TI-89, TI-92, or TI-92 Plus; *Casio* CFX-9850Ga-Plus, CFX-9850G, CFX-9800G, or FX-9700 series; or *Sharp* EL-9600c, EL-9600, EL-9300, or EL-9200. **Note:** Some of these calculators are no longer commercially manufactured.
- a computer and a spreadsheet program

  Any of the following spreadsheet programs may be used: *ClarisWorks™ 4.0* (or higher) for Macintosh, *ClarisWorks™ 4.0* (or higher) for Windows, *Microsoft® Works 3.0* (or higher) for Windows, or *Microsoft® Access 97* and *Microsoft® Excel 97* for Windows (a part of *Microsoft® Office Professional 95* or *Microsoft® Office Professional 97*).

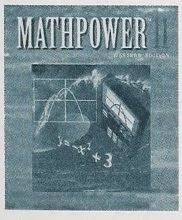
**Note:** Examples in this course show the TI-83 graphing calculator and the *ClarisWorks™ 4.0* spreadsheet program where applicable. If you have access to a videocassette recorder, you may wish to view the video *The TI-83 Graphing Calculator Video Tutor* to review some of the calculator's features. Check with your distance learning provider concerning the availability of this video.

Finally, you will need the following resources, which can be purchased locally:

- a mathematical instrument set (compass, protractor, ruler, and triangles)
- a binder in which to respond to the questions asked
- a binder for journal writing

# Other Visual Cues

In addition to the Mathematical Process icon described earlier, you will find other visual cues throughout this course. Read the explanations given to discover what each icon prompts you to do.



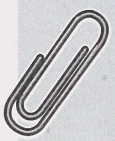

Refer to the textbook.

Use the Internet to explore a topic.

Answer the questions in the Assignment Booklet.

***Remember:*** *Any Internet website address given in this module is subject to change.*

# CONTENTS



## Module Overview


Evaluation
2
Strategies for Completing a Module
3


## Section 1

### Polynomial Functions


Activity 1: The Remainder Theorem
6
Activity 2: The Factor Theorem
13
Activity 3: Solving Polynomial Equations
21
Activity 4: Polynomial Functions and Inequalities
26
Follow-up Activities
33
Extra Help
33
Enrichment
37
Conclusion
43
Assignment
43


## Section 2

### The Algebra of Functions


Activity 1: Operations with Functions
46
Activity 2: Composition of Functions
53
Activity 3: Inverse Functions
58
Follow-up Activities
64
Extra Help
64
Enrichment
66
Conclusion
69
Assignment
69

## Section 3

**Absolute Value, Rational, and Radical Functions**


Activity 1: Absolute Value Functions, Equations, and Inequalities
72

Activity 2: Rational Functions, Equations, and Inequalities
81

Activity 3: Radical Functions, Equations, and Inequalities
88

Follow-up Activities
93

Extra Help
93

Enrichment
94

Conclusion
95

Assignment
95

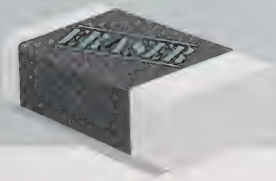

## Module Summary


Final Module Assignment
96


## Appendix


Glossary
98

Suggested Answers
99

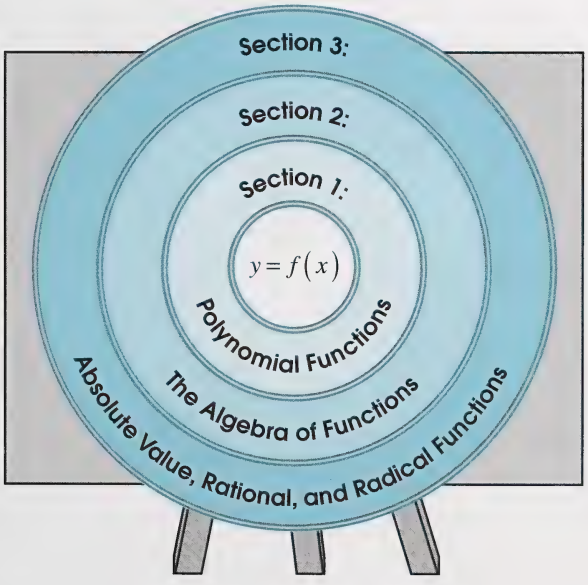
# Module Overview

Have you ever watched skilled archers focus on a target? Before they shoot, they shut out everything around them not central to their task. For a few moments, the target becomes the most important thing in their lives, their faces are steeped in concentration, and they do not take their eyes off the target until they have hit their mark.

The target of this module is the function. The notion of function is central to mathematics. It is a core principle for much of what you will study in mathematics.

Your goal in this module is to build on your knowledge of quadratic functions and explore polynomial functions and their graphs. You will investigate how functions are interrelated. You will then explore absolute value functions, rational functions, and radical functions; and you will use these functions to solve related equations and inequalities and real-world problems.

## Module 4: Functions



# Evaluation

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Functions, has three sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and practise or apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

At several points in this module you will be directed to the accompanying Assignment Booklet. In this module you are expected to complete three section assignments and one final module assignment. Your grading in this module is based upon the assignments that you submit for evaluation. The mark distribution is as follows:

| | |
|---|---|
| **Section 1 Assignment** | **30 marks** |
| **Section 2 Assignment** | **25 marks** |
| **Section 3 Assignment** | **30 marks** |
| **Final Module Assignment** | **15 marks** |
| **TOTAL** | **100 marks** |

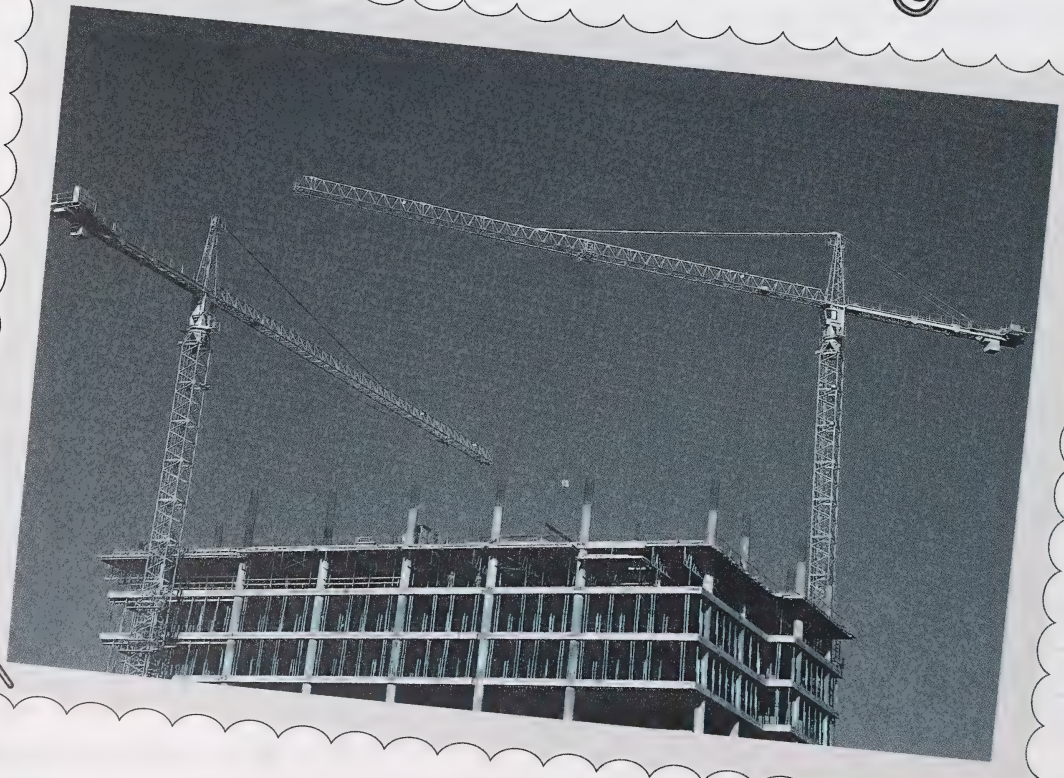
# Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklet, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 20.

Good luck!

# Section 1



# Polynomial Functions

Tower cranes (or overhead cranes), like the ones in the photograph, are common sights in commercial construction today. Tower cranes are mostly used in the construction of high-rise buildings. Not only do these cranes raise beams, walls, and other construction materials to enormous heights, they also help workers position these items in specified places. How do workers remove tower cranes once the work is done? Do you ever wonder how the crane that helped build the CN Tower was removed?

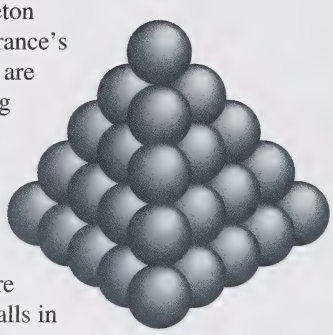
Like the office tower under construction, Section 1 deals with a part of mathematics that is still under construction—polynomial functions. This section will build on your previous knowledge of two types of polynomial functions: linear functions and quadratic functions.

In this section, you will investigate linear and quadratic functions. You will then analyse these functions using the Remainder Theorem and the Factor Theorem. Finally, you will solve equations and problems that contain polynomials and graph polynomial functions using a combination of technology and analytical skills.

# Activity 1: The Remainder Theorem

**Mathematical Process**
- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Have you visited the Fortress of Louisbourg on Cape Breton Island? Completed in 1745, it was built to help protect France's interests in the New World. Many visitors to Louisbourg are impressed by the cannons lining the stone ramparts facing the harbour.



Did you know cannonballs can be stacked in the shape of a triangular prism? There is one ball at the very top of the pile. In the second layer from the top, there are three cannonballs; in the third layer from the top, there are six cannonballs; and so on. The total number of cannonballs in the pile depends on the number of layers in the pile.

| Number of Layers | Number in the Bottom Layer | Total Number of Cannonballs |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 3 | 4 |
| 3 | 6 | 10 |
| 4 | 10 | 20 |
| 5 | 15 | 35 |
| $x$ | $\frac{x(x+1)}{2}$ | $\frac{x(x+1)(x+2)}{6}$ |

In the preceding table, there are three sequences. The first sequence is 1, 2, 3, …, $x$, …; it can be written in functional notation as $f(x)=x$, where $x \in N$. This is a linear function. The second sequence is 1, 3, 6, 10, …, $\frac{x(x+1)}{2}$, …; it can be written as $g(x)=\frac{x(x+1)}{2}$, where $x \in N$, or $g(x)=\frac{1}{2}x^2+\frac{1}{2}x$, where $x \in N$. This is a quadratic function. The third sequence is 1, 4, 10, 20, 35, …, $\frac{x(x+1)(x+2)}{6}$, …; it can be written as $h(x)=\frac{x(x+1)(x+2)}{6}$, where $x \in N$, or $h(x)=\frac{1x^3}{6}+\frac{1x^2}{2}+\frac{1x}{3}$, where $x \in N$. This is a **cubic function**.

*cubic function: a polynomial function of degree 3*

**1.** Use $h(x)=\frac{1}{6}x^3+\frac{1}{2}x^2+\frac{1}{3}x$, where $x \in N$, to determine the total number of cannonballs in

**a.** 5 layers **b.** 10 layers

---

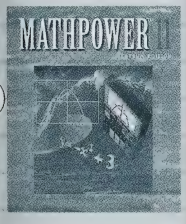
Compare your responses with those in the Appendix, Section 1: Activity 1.

---

Linear, quadratic, and cubic functions are all examples of **polynomial functions**.

Recall that a polynomial function is a function of the form $y=a_n x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1 x+a_0$, where $n \in W$.

In this activity, you will analyse polynomial functions using a theorem based on division by binomials.




Turn to page 198 of *MATHPOWER™ 11* and read "The Remainder Theorem."

**2.** Answer the following questions on page 198 of the textbook:

**a.** questions 1 to 6 of "Explore: Discover the Relationship"
**b.** questions 1 to 5 of "Inquire"

---

Compare your responses with those in the Appendix, Section 1: Activity 1.

---

You have just discovered that for a polynomial function $y=P(x)$, the value of $P(b)$ may be found by direct substitution or from the remainder when $P(x)$ is divided by $x-b$. This result is summarized by the **Remainder Theorem**.

**Remainder Theorem**

*When polynomial $P(x)$ is divided by $x-b$, the remainder equals $P(b)$.*

There would seem to be no apparent advantage in evaluating $P(b)$ by dividing by $x-b$ unless the division process can be made simpler. In the next example, you will explore a simpler method of dividing a polynomial by a binomial of the form $x-b$. This process is called **synthetic division**.

*synthetic division: a simpler, quicker method of dividing a polynomial by a binomial of the form $x-b$*

## Example

Given $P(x)=3x^3-5x^2+6x-7$, determine $P(2)$ using the following:

**a.** direct substitution
**b.** long division and the Remainder Theorem
**c.** synthetic division and the Remainder Theorem

## Solution

**a.** $P(x)=3x^3-5x^2+6x-7$

$$\begin{aligned}P(2)&=3(2)^3-5(2)^2+6(2)-7\\&=3(8)-5(4)+12-7\\&=24-20+12-7\\&=9\end{aligned}$$

**b.** The Remainder Theorem states that $P(2)$ equals the remainder when $P(x)$ is divided by $x-2$.

$$\begin{array}{r}3x^2+x+8\\ x-2\overline{)\,3x^3-5x^2+6x-7}\\ \underline{3x^3-6x^2}\\ x^2+6x\\ \underline{x^2-2x}\\ 8x-7\\ \underline{8x-16}\\ 9\end{array}$$

Because the remainder is 9, $P(2)=9$.

**c.** To use synthetic division, follow these steps:

**Step 1:** Arrange the terms of $P(x)$ in descending degree. List the coefficients of the terms in the same order.

Since the terms of $P(x)=3x^3-5x^2+6x-7$ are already in descending order, the coefficients should be listed as follows:

3 −5 6 −7

**Step 2:** Identify the value of $b$ for your divisor and position it in front of the list of coefficients (as shown).

$$\begin{array}{r|llll} 2 & 3 & -5 & 6 & -7 \\ \hline & & & & \end{array}$$

Because $x-b=x-2$, the value of $b$ is 2.

**Step 3:** Rewrite the first coefficient below the horizontal line.

$$\begin{array}{r|llll} 2 & 3 & -5 & 6 & -7 \\ \hline & 3 & & & \end{array}$$

**Step 4:** Multiply 3 by 2; then place the product, 6, between $-5$ and the horizontal line.

$$\begin{array}{r|llll} 2 & 3 & -5 & 6 & -7 \\ & & 6 & & \\ \hline & 3 & & & \end{array}$$

**Step 5:** Add $-5$ and 6; then place the sum, 1, below the horizontal line.

$$\begin{array}{r|llll} 2 & 3 & -5 & 6 & -7 \\ & & 6 & & \\ \hline & 3 & 1 & & \end{array}$$

**Step 6:** Multiply 1 by 2; then place the product, 2, between 6 and the horizontal line.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −5 | 6 | −7 |
| | | 6 | 2 | |
| | 3 | 1 | | |

**Step 7:** Add 6 and 2; then place the sum, 8, below the horizontal line.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −5 | 6 | −7 |
| | | 6 | 2 | |
| | 3 | 1 | 8 | |

**Step 8:** Multiply 8 by 2; then place the product, 16, between $-7$ and the horizontal line.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −5 | 6 | −7 |
| | | 6 | 2 | 16 |
| | 3 | 1 | 8 | |

**Step 9:** Add $-7$ and 16; then place the sum, 9, below the horizontal line.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −5 | 6 | −7 |
| | | 6 | 2 | 16 |
| | 3 | 1 | 8 | 9 ← This is the remainder. |

This is the quotient $3x^2+x+8$.

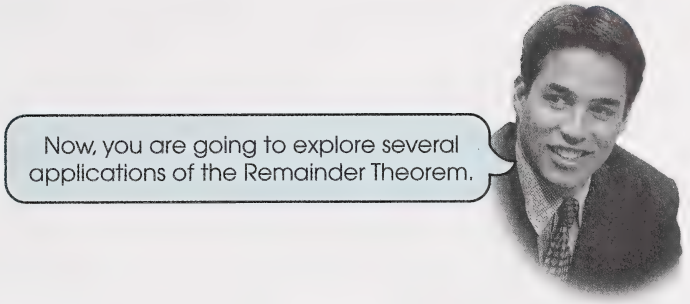
Because the remainder is 9, $P(2)=9$.

**3.** Use synthetic division to find the quotient and remainder for each of the following:

**a.** $\left(x^3-x+7\right)\div(x+2)$ **b.** $\left(4x^2-7\right)\div(x-1)$

**4.** Use synthetic division and the Remainder Theorem to determine the total number of cannonballs in a pile 3 layers high.

Compare your responses with those in the Appendix, Section 1: Activity 1.




Turn to page 199 of *MATHPOWER™ 11* and read from the top of the page to the bottom of page 200, working through Examples 1 to 3.

**5.** Answer the following questions on pages 202 and 203 of the textbook:

**a.** questions 8, 16, 22, 26, 30, 39, 47, 50, 51, 52, and 53 of "Practice"
**b.** questions 55 and 58 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 1.

Recall that the Remainder Theorem states, "When the polynomial $P(x)$ is divided by $x-b$, the remainder equals $P(b)$." But suppose you were dividing by a binomial of the form $ax-b$, where $a \neq 1$. How is the remainder related to $ax-b$?

Turn to page 201 of *MATHPOWER™ 11* and read from the top of the page to the red line on page 202, working through Examples 4 and 5.

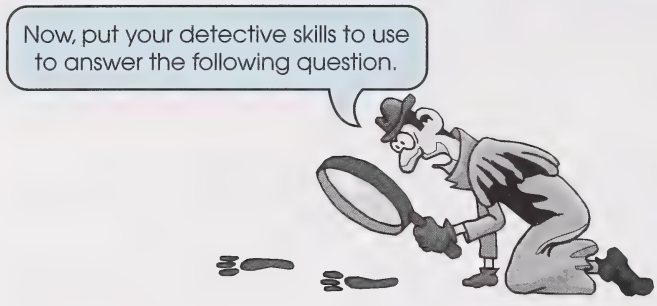
**6.** Answer the following questions on pages 202 and 203 of the textbook:

**a.** questions 41 and 42 of "Practice"
**b.** questions 54, 62, and 63.a. of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 1.

## Now Try This

A good problem solver in mathematics is like a good detective. A good detective must look for the facts and analyse the situation carefully, must not jump to conclusions, and must be prepared to consider alternatives.



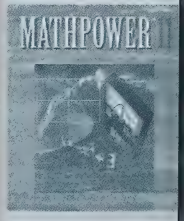

**7.** Turn to "Problem Solving" on page 235 of *MATHPOWER™ 11* and answer question 6 of "Using the Strategies."

Compare your response with the one in the Appendix, Section 1: Activity 1.

## Looking Back

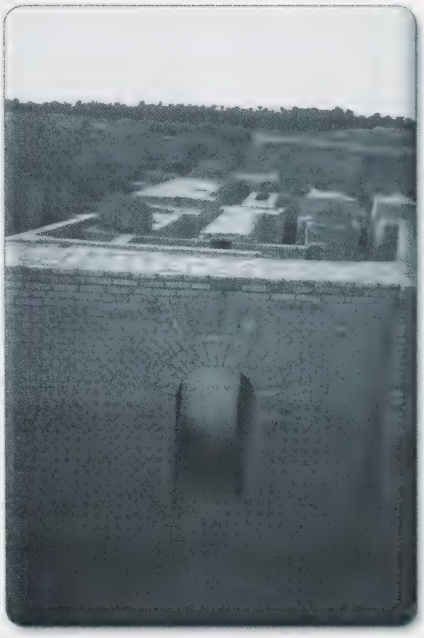
In this activity, you looked at relationships between a polynomial and divisors of the form $x - b$ and $ax - b$. In your journal, summarize these relationships and support each statement with an example. Give your journal to another student taking Pure Mathematics 20 and have him or her read your summary and critique it for clarity and accuracy.

**Mathematical Process**
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization




**Mathematical Process**
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

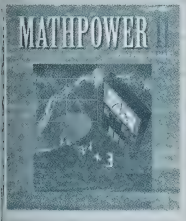
# Activity 2: The Factor Theorem



Five thousand years ago between the Tigris and Euphrates rivers (in present-day Iraq), part of the region often called the "cradle of Western civilization," the Babylonians used a number system based on 60. This system continues today in angle measurement and in divisions of time. You can thank ancient Babylonian scribes and accountants for 360 degrees in a circle, 60 seconds in a minute, and 60 minutes in an hour.

One of the reasons suggested for using 60 as base is its multiple factors; 60 can be divided evenly by 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. As a result, work with fractional parts was made much simpler.

The concept of factor continues to play an important role in mathematics today. In this activity, you will investigate the factors of polynomials and procedures for finding these factors.




Turn to page 204 of *MATHPOWER™ 11* and read "The Factor Theorem."

**1.** Answer the following questions on pages 204 and 205 of the textbook:

**a.** questions 1 to 12 of "Explore: Discover the Relationship"
**b.** questions 1 to 4 of "Inquire"

---

Compare your responses with those in the Appendix, Section 1: Activity 2.

---

The results of your investigation are summarized by the **Factor Theorem**.

**Factor Theorem**

*If* $x-b$ *is a factor of polynomial* $P(x)$, *then* $P(b)=0$. *Conversely, if* $P(b)=0$, *then* $x-b$ *is a factor of polynomial* $P(x)$.

Another result relates the zero, $b$, obtained from the factor, $x-b$, to the constant term in the polynomial, $P(x)$.

## Example

Given $P(x)=(x-2)(x+3)(x-5)$,

**a.** write the polynomial in expanded form

**b.** identify the zeros of the polynomial

**c.** explain how the zeros are related to the constant of the polynomial in expanded form

## Solution

**a.** $P(x)=(x-2)(x+3)(x-5)$

$=(x-2)\left(x^2-5x+3x-15\right)$

$=(x-2)\left(x^2-2x-15\right)$

$=x^3-2x^2-15x-2x^2+4x+30$

$=x^3-4x^2-11x+30$

**b.** The zeros are obtained directly from the factors.

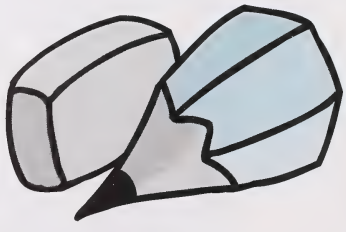
$x-2=0$ $\quad$ $x+3=0$ $\quad$ $x-5=0$

$x=2$ $\quad$ $x=-3$ $\quad$ $x=5$

Therefore, the zeros are 2, $-3$, and 5.

**c.** The integral zeros of this polynomial are all factors of the constant term, 30. This makes sense, since the zeros were obtained from the second term of each factor. The product of the second terms of the factors is the constant term of the polynomial.

$(-2)(+3)(-5)=30$



The preceding example illustrates the **Integral Zero Theorem**.

**Integral Zero Theorem**

*If $x = b$ is an integral zero of polynomial $P(x)$ with integral coefficients, then b is a factor of the constant term.*

You will now apply the Factor Theorem and the Integral Zero Theorem.

Turn to page 205 of *MATHPOWER™ 11* and read from the red line to the bottom of page 207, working through Examples 1 to 4.

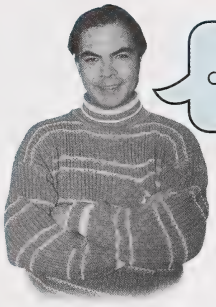
**2.** Answer the following questions on pages 209 to 211 of the textbook:

**a.** questions 1, 3, 7, 8, 22, 24, 26, 55, 58, and 61 of "Practice"
**b.** questions 75, 78.a., 82.a., 85, and 86 of "Applications and Problem Solving"

---

Compare your responses with those in the Appendix, Section 1: Activity 2.

---



The real zeros of a polynomial function, $y = P(x)$, can be approximated using a graphing calculator. Consider the following example.

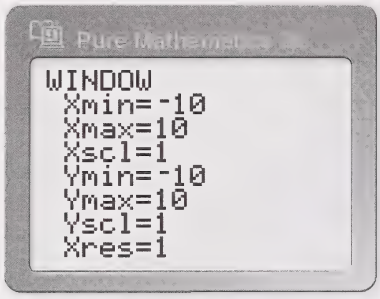
## Example

**a.** Graph $P(x) = x^3 + 2x^2 - 5x - 6$ on a graphing calculator.
**b.** Use the Zero feature of your graphing calculator to approximate the zeros.
**c.** Write the possible factors of $P(x)$.
**d.** Verify that your factors are correct.

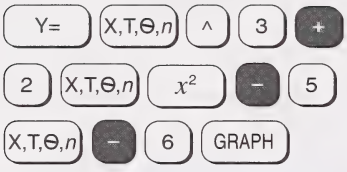
**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

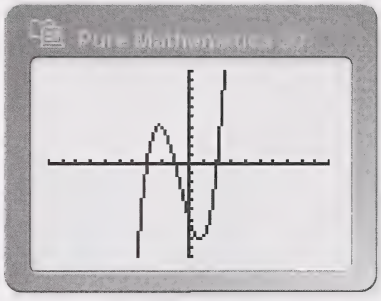
## Solution

**a.** Use the standard window settings to graph the polynomial.



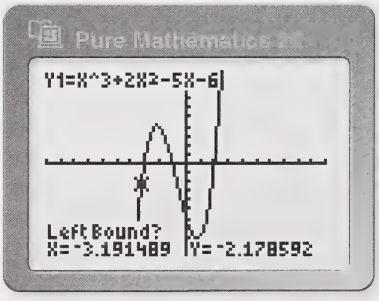

Press the following to graph the polynomial:



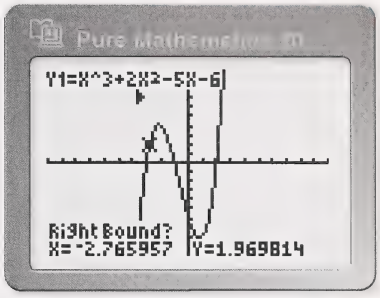



**b.** The $x$-intercepts of the graph appear to be $-3, -1$, and 2. To check these approximations using the Zero feature, press 2nd [ CALC ] 2.

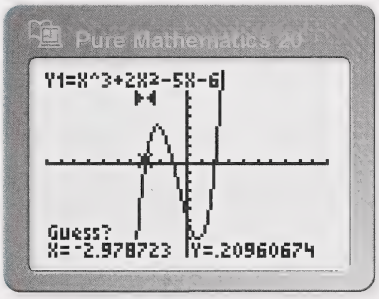
The cursor is positioned on the $y$-axis. To check the $x$-intercept on the far left, use the arrow keys to move the cursor along the curve until it lies to the left of this $x$-intercept. The screen prompts you to enter a left bound; so, press ENTER.

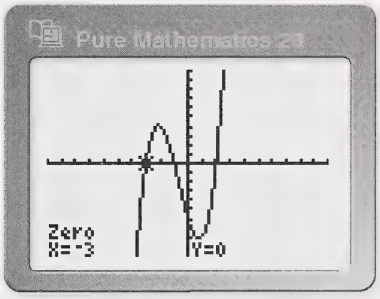


You are now prompted to enter a right bound. Use the arrow keys to position the cursor to the right of the $x$-intercept, and press ENTER.




Lastly, you are prompted to guess the zero. Simply press the arrow key once to move the cursor to the left, and press ENTER.




The calculator indicates that $-3$ is the $x$-intercept on the far left. This is the first zero of the function.

Following a similar sequence of steps, use your graphing calculator to confirm that the other two zeros are $-1$ and 2.

**c.** Determine the factors from the zeros.

The possible factorization of $y = P(x)$ is $x^3 + 2x^2 - 5x - 6 = (x+3)(x+1)(x-2)$.

| Zero | Factor |
|---|---|
| $-3$ | $x+3$ |
| $-1$ | $x+1$ |
| 2 | $x-2$ |

**d.** Verify.

| LS | RS |
|---|---|
| $x^3+2x^2-5x-6$ | $(x+3)(x+1)(x-2)$ |
| | $=(x+3)\left(x^2-2x+x-2\right)$ |
| | $=(x+3)\left(x^2-x-2\right)$ |
| | $=x^3-x^2-2x+3x^2-3x-6$ |
| | $=x^3+2x^2-5x-6$ |

LS = RS

$\therefore\ x^3+2x^2-5x-6=(x+3)(x+1)(x-2)$

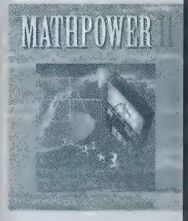
All of the questions you have worked with so far have had integral zeros obtained from factors of the form $x-b$. What adjustments are necessary if the factor is $ax-b$, where $a\neq 1$, and the corresponding zero is $\frac{b}{a}$?

Turn to page 208 of *MATHPOWER™ 11* and read from the top of the page to the red line, working through Example 5.

**3.** Answer questions 43, 48, and 49 of "Practice" on page 210 of the textbook.

Compare your responses with those in the Appendix, Section 1: Activity 2.

It's possible that a polynomial with integral coefficients has no integral zeros. How would you test for rational zeros? What values would you try?



## Example

How are the zeros of $P(x)=6x^2-x-35$ related to its coefficients?

### Solution

Obtain the zeros of the polynomial from its factors.

$$\begin{aligned} P(x) &= 6x^2-x-35 \\ &= (2x-5)(3x+7) \end{aligned}$$

$$\begin{aligned} \therefore\ 2x-5 &= 0 \quad \text{or} \quad 3x+7=0 \\ 2x &= 5 \qquad\qquad 3x=-7 \\ x &= \frac{5}{2} \qquad\qquad x=\frac{-7}{3} \end{aligned}$$

The zeros of the polynomial are $\frac{5}{2}$ and $\frac{-7}{3}$.

The zeros are related to the coefficients of $P(x)$ as follows:

- The numerators of the rational zeros are factors of the constant term.

  $5\times(-7)=-35$

- The denominators of the rational zeros are factors of the coefficient of the highest-degree term.

  $2\times 3=6$

The **Rational Zero Theorem** summarizes the results of this example.

#### Rational Zero Theorem

*If $\frac{b}{a}$ is a rational zero of polynomial $P(x)$ with integral coefficients, then b is a factor of the polynomial's constant term and a is a factor of the coefficient of the highest-degree term.*

Turn to page 208 of *MATHPOWER™ 11* and read from the red line to the red line on page 209, working through Example 6.

**4.** Answer the following questions on pages 210 and 211 of the textbook:

**a.** questions 65 and 74 of "Practice"
**b.** questions 79.a., 84, 89, and 93 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 2.

## Now Try This

In this activity, you have worked with zeros of cubic functions and **quartic functions**. The zeros of a polynomial function correspond to the $x$-intercepts of its graph. You will now use your graphing calculator to explore the relationship between the zeros of a polynomial and its degree.

Turn to "TECHNOLOGY" on page 212 of *MATHPOWER™ 11* and read "Exploring Polynomial Functions With a Graphing Calculator."

**5.** Answer the following questions on pages 212 and 213 of the textbook:

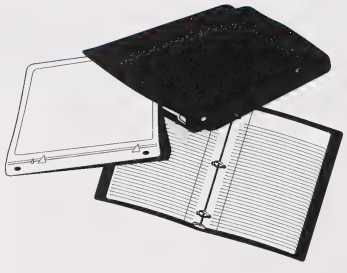
**a.** questions 1 to 9 of Investigation 1, "Cubic or Third-Degree Functions and Equations"

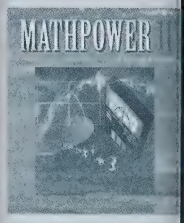
**b.** questions 1 to 9 of Investigation 2, "Quartic or Fourth-Degree Functions and Equations"

Compare your responses with those in the Appendix, Section 1: Activity 2.

## Looking Back

In this activity, you explored procedures for finding zeros of polynomial functions with integral coefficients. In your journal, summarize and provide examples of all the mathematical theorems you studied in this activity.






*quartic function: a polynomial function of degree 4*

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

# Activity 3: Solving Polynomial Equations

CORBIS/CRAIG AURNESS

**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Did you ever wonder who constructed Britain's Stonehenge and what its purpose was? Modern techniques date construction over a 1500-year period, beginning 5000 years ago. The original builder constructed a large circular earthwork or henge, consisting of a bank and a ditch. An inner circle of 80 bluestones was added 1000 years later. These stones, approximately 4 t each, are thought to have been quarried in the Prescelly Mountains of Wales 400 km away. The outer ring of giant sarsen stones, hauled from a site 30 km north of Stonehenge and weighing as much as 50 t apiece, was completed 3500 years ago. These stones were capped with horizontal lintel stones completing the ring.

If you have access to the Internet, you can obtain more information about Stonehenge at the following website:

**http://www.amherst.edu/~ermace/sth/sth.html**

The volume (in $m^3$) above the ground of each of the giant sarsen stones can be represented by the polynomial expression $x^3 + 4x^2 + 3x$, where $x$, $x+1$, and $x+3$ are dimensions of the rectangular prism.

$$V = x(x+1)(x+3)$$
$$= x^3 + 4x^2 + 3x$$

If the volume is 8 $m^3$, the value of $x$ can be found by solving the **polynomial equation** $x^3 + 4x^2 + 3x - 8 = 0$. In this activity, you will explore methods for solving polynomial equations, mostly involving **cubic equations**.

*polynomial equation: the equation formed when a polynomial is set equal to 0*

*cubic equation: a polynomial equation of degree 3*

Turn to page 214 of *MATHPOWER™ 11* and read "Solving Polynomial Equations."

**1.** Answer the following questions on page 214 of the textbook:

**a.** questions a. to d. of "Explore: Interpret the Graph"
**b.** questions 1 and 2 of "Inquire"

Compare your responses with those in the Appendix, Section 1: Activity 3.

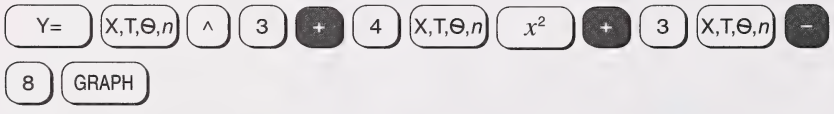
In the preceding exercises, you discovered that the roots of polynomial equations are the zeros of corresponding polynomial functions (the $x$-intercepts of the graphs of these functions). You can use your graphing calculator to graph these functions and approximate their zeros, thereby obtaining approximations of the roots of the original equations.

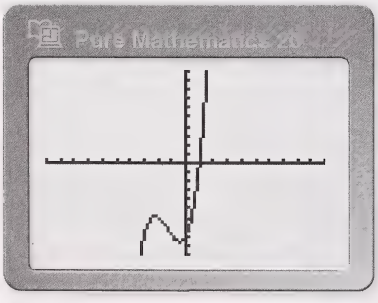
## Example

Determine the value of $x$ that approximates the smallest dimension of the giant sarsen stones in Stonehenge. This value is a root of the polynomial equation $x^3 + 4x^2 + 3x - 8 = 0$.

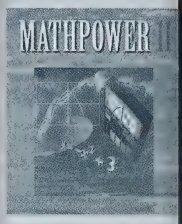
### Solution

Graph the corresponding polynomial function $y = x^3 + 4x^2 + 3x - 8$ on your graphing calculator using the standard window settings.

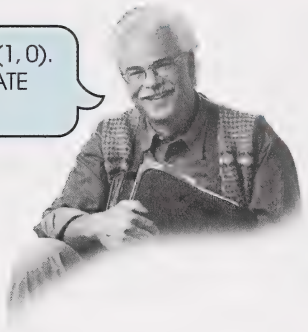



*Note:* There is only one $x$-intercept. Therefore, there is only one real zero.




**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization




**Check**

Verify your answer by substituting $x = 1$ into the original cubic equation.

| LS | RS |
|---|---|
| $x^3 + 4x^2 + 3x - 8$ | 0 |
| $= 1^3 + 4(1)^2 + 3(1) - 8$ | |
| $= 1 + 4 + 3 - 8$ | |
| $= 0$ | |

LS = RS

Therefore, $x = 1$ is the real root of the equation $x^3 + 4x^2 + 3x - 8 = 0$.

Thus, the shortest dimension of the sarsen stone is about 1 m.

Another method of solving polynomial equations is by factoring.

Turn to page 215 of *MATHPOWER™ 11* and read from the top of the page to the bottom of page 216, working through Examples 1 to 3.

**2.** Answer the following questions on pages 219 to 220 of the textbook:

**a.** questions 4, 5, 9, 11, 13, 27, 35, and 42 of "Practice"

**b.** questions 81.a., 82, 83.a., and 86 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 3.

The roots of a polynomial equation can be real or non-real. Real roots can be integers, rationals, or irrationals. In the next part of this activity, you will look at irrational roots and non-real roots.

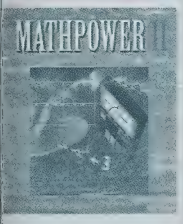
Turn to pages 217 to 219 of *MATHPOWER™ 11* and work through Examples 4 to 6.

**3.** Answer the following questions on pages 220 to 221 of the textbook:

**a.** questions 48, 51, 60, and 65 of "Practice"

**b.** questions 75, 80, 93, 100, and 104 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 3.



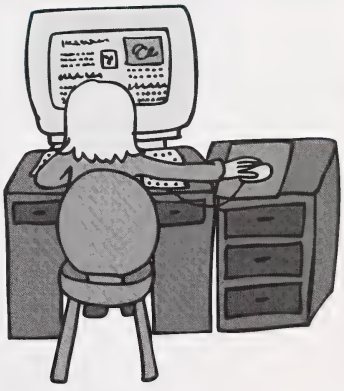

## Now Try This

Not all problems that are easy to pose are easy to answer. A problem that took mathematicians centuries to solve is Fermat's Last Theorem. In 1630, Fermat conjectured that $x^n + y^n = z^n$ has no non-zero integral solutions for $x$, $y$, and $z$ when $n > 2$. The first proof of this theorem didn't appear until 1993.

Now, put your logic skills to use to answer the following problem. You should be able to solve it more quickly than Fermat's Last Theorem.

**4.** Turn to page 221 of *MATHPOWER™ 11* and answer "LOGIC POWER."

Compare your response with the one in the Appendix, Section 1: Activity 3.



Now that you have solved this problem, you may want to research Fermat's Last Theorem—a problem that perplexed mathematicians for over 360 years. The story of the search for its solution stretches from the year 1630, when it is thought Fermat first penned the statement of his theorem as a margin note in a text he was studying, to the year 1993, when a British mathematician by the name of Andrew Wiles completed the proof of the theorem while working in the United States.

If you have access to the Internet, you can read about this intriguing search at the following website:

**http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Fermat's_last_theorem.html**

## Looking Back

In this activity, you solved polynomial equations by graphing, by factoring, and by using the quadratic formula to obtain irrational or non-real roots of quadratic factors.

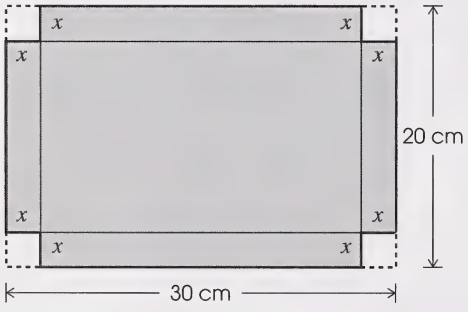
In your journal, make up a practice test based on the concepts in this activity. Also, prepare an answer key that you could use to mark the test. Exchange tests with a friend who is also taking Pure Mathematics 20. Write the test, then mark each other's work. Finally, together with your friend, identify areas where mistakes are commonly made.

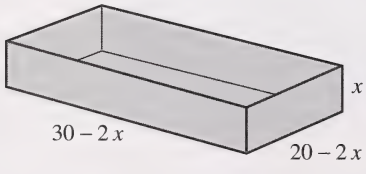
**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

# Activity 4: Polynomial Functions and Inequalities

What kind of shoes did the customer purchase? Do you think she bought a pair of runners or a pair of dress shoes? Is it possible to tell just by looking at the box? There are many considerations in box design. A few considerations when designing a box are the size and shape of its intended contents, the appearance and product recognition, sturdiness, the cost to the manufacturer, and its other potential uses.

Suppose you are given a rectangular sheet of cardboard 20 cm × 30 cm. Using the cardboard, you are asked to construct an open box by cutting equal squares from the corners and turning up the tabs (as shown).




**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

If the width of each tab is $x$, then the volume, $V$, of the box is

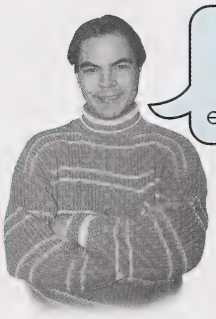
$$\begin{aligned} V &= \ell \times w \times h \\ &= (30-2x)(20-2x)x \\ &= 4x^3 - 100x^2 + 600x \end{aligned}$$

If the box is to hold at least 1000 $cm^3$, what should $x$ be?

$$\begin{aligned} V &\geq 1000 \\ 4x^3 - 100x^2 + 600x &\geq 1000 \\ 4x^3 - 100x^2 + 600x - 1000 &\geq 0 \end{aligned}$$

The possible values of $x$ are the solutions to this **polynomial inequality**.

*polynomial inequality: an inequality containing a polynomial expression*



In this activity, you will investigate the graphs of polynomial functions and use those graphs to solve polynomial inequalities. You will begin by exploring polynomial functions and their graphs.

Turn to page 274 of *MATHPOWER™ 11* and read "Polynomial Functions and Inequalities."

**1.** Answer the following questions on page 274 of the textbook:

**a.** "Explore: Draw a Graph"
**b.** questions 1 to 5 of "Inquire"

---

Compare your responses with those in the Appendix, Section 1: Activity 4.

---

In Activities 1 to 3, you worked with several different types of polynomial functions: constant, linear, quadratic, cubic, and quartic functions. These functions are grouped according to degree.

Turn to page 274 of *MATHPOWER™ 11* and read from the red line to the red line on page 275, working through Example 1.

**2.** Answer questions 1 to 5 of "Practice" on page 282 of the textbook.

---

Compare your responses with those in the Appendix, Section 1: Activity 4.

---

There are several techniques you can use to graph polynomial functions. In this activity, you will use tables of values, zeros, and your graphing calculator. From the graph, you will describe the polynomial function in many of the same ways you described quadratic functions. In the next set of examples, you will explore these techniques.

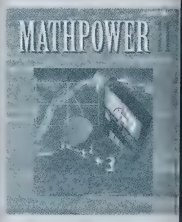
Turn to page 275 of *MATHPOWER™ 11* and read from the red line to the red line on page 278, working through Examples 2 to 5.

**3.** Answer questions 10, 17, 23, 29, and 30 of "Practice" on pages 282 and 283 of the textbook.

---

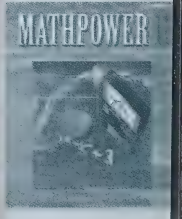
Compare your responses with those in the Appendix, Section 1: Activity 4.

---




**Mathematical Process**

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization



So far, you have focussed on the zeros of polynomial functions and the roots of polynomial equations. You will now use these skills to help you solve polynomial inequalities.

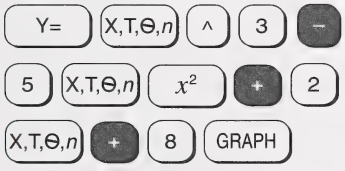
## Example

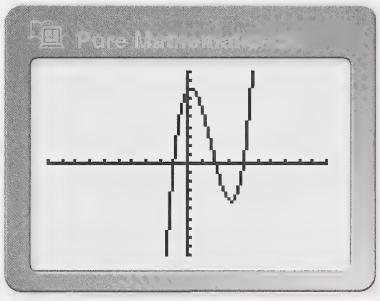
Solve $x^3 - 5x^2 + 2x + 8 > 0$

**a.** using a graphing calculator
**b.** using test intervals

## Solution

**a.** Use your graphing calculator to graph the corresponding polynomial function $f(x) = x^3 - 5x^2 + 2x + 8$. You will use this graph to determine the values of $x$ for which $f(x) > 0$; that is, for which $x^3 - 5x^2 + 2x + 8 > 0$.




The zeros of the function are $-1$, 2, and 4.

The zeros can be obtained using a table of values or by using the Zero feature.

Recall that the zeros of a function are the $x$-intercepts of its graph. Look at the portion of the graph between the $x$-intercepts $-1$ and 2. The graph lies above the $x$-axis in this interval. Therefore, $f(x) > 0$ when $-1 < x < 2$. The graph also lies above the $x$-axis for all values of $x$ to the right of the $x$-intercept 4. Therefore, $f(x) > 0$ when $x > 4$.

Thus, the solution to $x^3 - 5x^2 + 2x + 8 > 0$ is $-1 < x < 2$ or $x > 4$.

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

**b.** Begin by finding the zeros or $x$-intercepts of the graph of function $f(x)=x^3-5x^2+2x+8$. As in part a., these zeros can be found from the factors.

The possible integral zeros are $\pm1, \pm2, \pm4$, and $\pm8$.

> **Remember:** *These possible zeros are obtained by listing the factors of the constant term, 8.*

Try $f(-1)$.

$$\begin{aligned} f(x)&=x^3-5x^2+2x+8 \\ f(-1)&=(-1)^3-5(-1)^2+2(-1)+8 \\ &=-1-5-2+8 \\ &=0 \end{aligned}$$

Therefore, $x+1$ is a factor.

Use synthetic division to find a second factor.

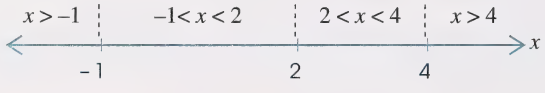
$$\begin{array}{r|cccc} -1 & 1 & -5 & 2 & 8 \\ & & -1 & 6 & -8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

Another factor is $x^2-6x+8$.

$$\begin{aligned} \therefore\ x^3-5x^2+2x+8&=(x+1)\left(x^2-6x+8\right) \\ &=(x+1)(x-2)(x-4) \end{aligned}$$

From the factors, the zeros are $-1$, 2, and 4.

These values divide the $x$-axis into four test intervals.



You can use the factors to determine the intervals for which $f(x) > 0$. Consider the following table:

| Interval | $x+1$ | $x-2$ | $x-4$ | $f(x)$ |
|---|---|---|---|---|
| $x < -1$ | − | − | − | $(-)(-)(-) = -$ |
| $-1 < x < 2$ | + | − | − | $(+)(-)(-) = +$ |
| $2 < x < 4$ | + | + | − | $(+)(+)(-) = -$ |
| $x > 4$ | + | + | + | $(+)(+)(+) = +$ |

To determine the signs of the factors that appear in the table, simply substitute representative values from the intervals into the factors to see if they are positive or negative. For example, to determine the sign of the factor $x+1$ when $x < -1$, pick a number, like $-2$, from the interval $x < -1$; then substitute it for $x$.

$$x+1 = -2+1$$
$$= -1 \quad \longleftarrow \text{This is less than 0.}$$

Therefore, $x+1 < 0$ when $x < -1$ (as shown by the negative sign in the table).

Because the original problem was to solve $x^3 - 5x^2 + 2x + 8 > 0$, the two intervals in the table for which $f(x) > 0$ are $-1 < x < 2$ and $x > 4$. Thus, the solution to $x^3 - 5x^2 + 2x + 8 > 0$ is $-1 < x < 2$ or $x > 4$.

The solution can be graphed on a number line.

Now, you will practise these procedures.

Turn to page 278 of *MATHPOWER™ 11* and read from the red line to the bottom of page 281, working through Examples 6 to 8.

**4.** Answer the following questions on pages 283 to 285 of the textbook:

**a.** questions 33, 41, 46, 49, and 54 of "Practice"
**b.** questions 57.c., 62, 64, and 70 of "Applications and Problem Solving"
**Note:** Use your graphing calculator to answer question 70.

Compare your responses with those in the Appendix, Section 1: Activity 4.

## Now Try This

Can you guess the contents of each box the woman in the photograph is carrying? Is the shape of a box always determined by its intended contents? Not always! Other considerations in box design, such as the following, are also determining factors:

- the size of its intended contents
- the shape of its intended contents
- the appearance
- product recognition
- sturdiness
- the cost of manufacturing
- other potential uses

In the introduction to this activity, you looked at the design of a box.

Reread the introduction; then answer the following question.

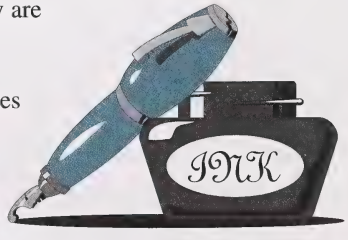
**5.** Solve the inequality that models the box problem, $4x^3 - 100x^2 + 600x - 1000 \geq 0$. Be certain to include any appropriate restrictions on $x$.

Compare your response with the one in the Appendix, Section 1: Activity 4.

## Looking Back

In this activity, you examined polynomial functions and their graphs. You then used the graphs of polynomial functions to help solve polynomial equations and inequalities. In the future, when you take other courses in mathematics, such as Mathematics 31, you will discover that the skills you mastered in this activity are fundamental.

In your journal, write about how the mathematics courses you will be taking in the future will help you achieve your career goals. Research which courses you might take in high school and in post-secondary institutions.




**Mathematical Process**
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

**Mathematical Process**
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

# Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

## Extra Help

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

### Example

**a.** Use your graphing calculator to graph $f(x) = x^3 - 3x + 2$.

**b.** Are there any real zeros? Round your answer to the nearest tenth if necessary.

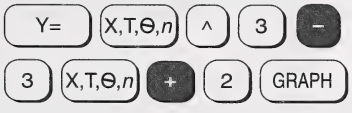
**c.** What is the $y$-intercept?

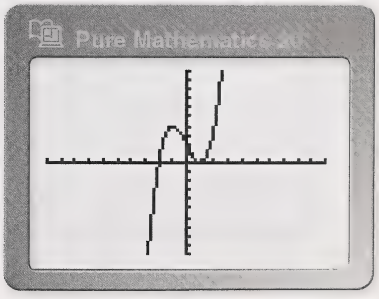
**d.** What are the approximate coordinates of any relative maximum or relative minimum?

**e.** Describe the end behaviour.

### Solution

**a.** Graph the function using the standard window settings.

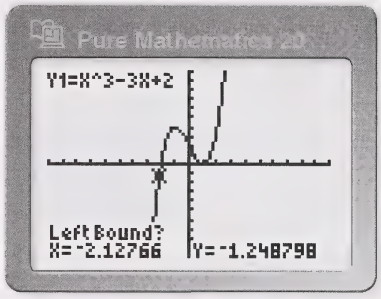


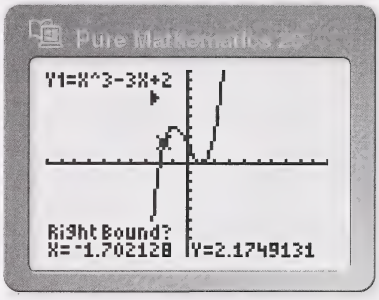
**b.** The graph appears to cross the $x$-axis at $(-2, 0)$ and $(1, 0)$. You can use the Zero feature to confirm these points.

To confirm $(-2, 0)$, start by pressing [2nd] [ CALC ] [2].

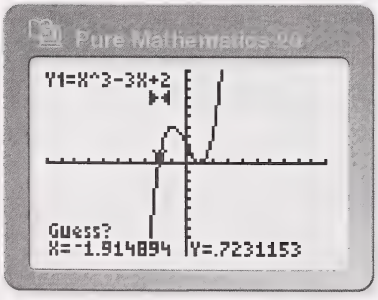
Next, use the arrow keys to position the cursor to the left of the $x$-intercept, and press [ENTER].

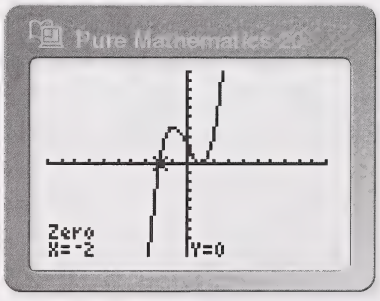


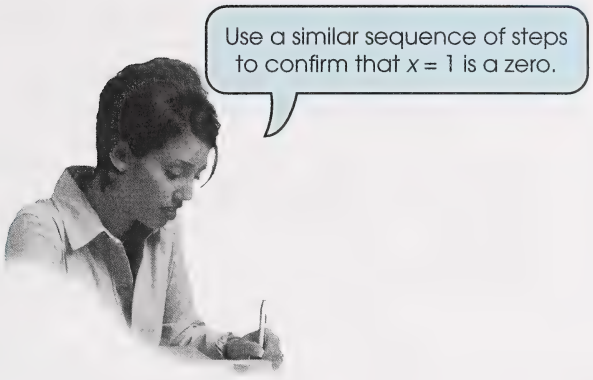

Now, use the arrow keys to position the cursor to the right of the $x$-intercept, and press [ENTER].




Lastly, enter your guess. Move the cursor one position to the left using the arrow key; then press [ENTER].






Because the graph appears to be tangent at $x=1$, the equation $f(x)=0$ must have two roots that equal 1; the factor $x-1$ is a squared factor. Therefore, the factors of the function are $f(x)=(x+2)(x-1)^2$.

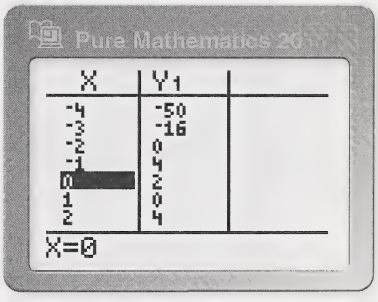
**Check**

$$\begin{aligned} f(x) &= (x+2)(x-1)^2 \\ &= (x+2)\left(x^2-2x+1\right) \\ &= x^3-3x+2 \end{aligned}$$

The zeros are $-2$ and 1.

**c.** From the graph, the $y$-intercept appears to be 2.

Use the table of values from your calculator to check this value. Press 2nd [ TABLE ].




The ordered pair $(0,2)$ shows that the $y$-intercept is 2.

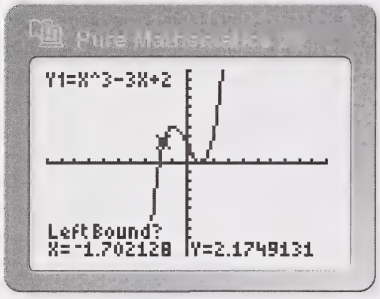
**Note:** The zeros $-2$ and 1 also appear in the tables as $(-2,0)$ and $(1,0)$.

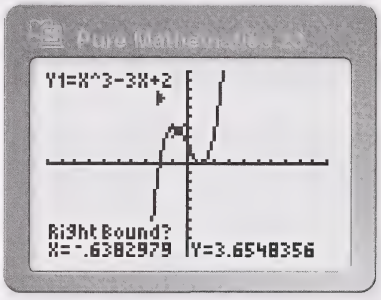
**d.** Use the Minimum and Maximum features from the CALCULATE menu to determine the relative minimum value and the relative maximum value.

On the graph, there appears to be a relative maximum at $x = -1$.

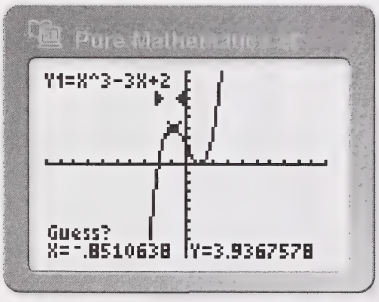
To confirm, start by pressing [2nd] [ CALC ] [4].

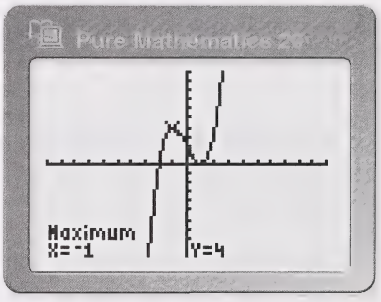
Next, locate a left bound using the arrow keys, and press [ENTER]. Locate a right bound using the arrow keys, and press [ENTER].




Now enter your guess by moving the cursor one position to the left and pressing [ENTER].




A relative maximum occurs at $(-1, 4)$.

Similarly, using the Minimum feature, a relative minimum occurs at $(1, 0)$.

**e.** The left-most $y$-values are negative, and the right-most $y$-values are positive.

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- [x] Visualization

For each of the following functions,

**a.** display its graph using your graphing calculator
**b.** determine whether there are any real zeros, rounding to the nearest tenth if necessary
**c.** determine the *y*-intercept
**d.** approximate the coordinates of any relative maximums or minimums
**e.** describe the end behaviours

**1.** $f(x)=x^3-7x+6$ **2.** $f(x)=-x^3+2x^2+4x-4$

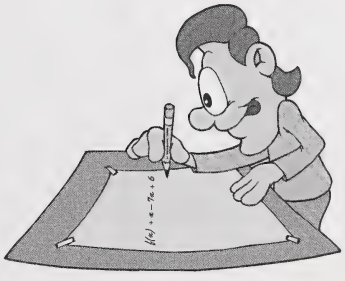
Compare your responses with those in the Appendix, Section 1: Follow-up Activities, Extra Help.

## Enrichment

Recall, from Module 3, that a formula exists for extracting roots of a cubic equation. This formula, called Cardano's Formula, is difficult to remember and even more difficult to apply using pencil and paper.

For example, you must first transform a cubic equation of the form $ax^3+bx^2+cx+d=0$ into a reduced cubic equation $y^3+py+q=0$, which does not contain a quadratic term. This reduction is obtained by replacing $x$ in the original equation by $y-\frac{b}{3a}$.

Once you have determined the reduced form, you must calculate $u$ and $v$, where $u=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$ and $v=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$, before the roots of the reduced equation can be found.



Now, roots $y_1$, $y_2$, and $y_3$ of the reduced equation are

$$y_1 = u + v$$

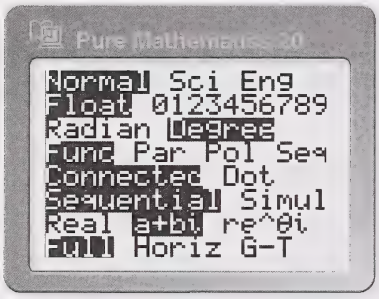
$$y_2 = -\frac{(u+v)}{2} + \frac{(u-v)i\sqrt{3}}{2}$$

$$y_3 = -\frac{(u+v)}{2} - \frac{(u-v)i\sqrt{3}}{2}$$

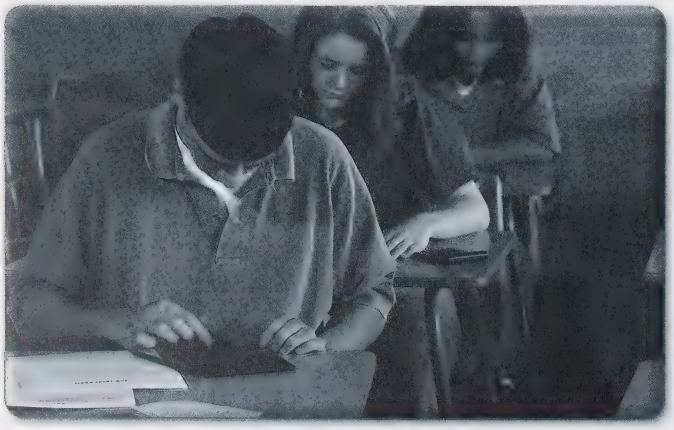
Finally, the roots of the original equation are obtained using the substitution $x = y - \frac{b}{3a}$.

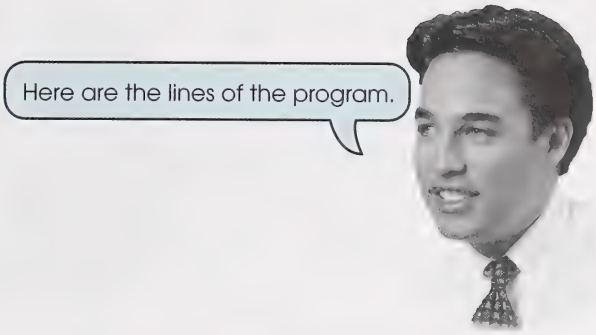
Fortunately, you can program your calculator. Once programmed, your calculator will give both real and imaginary roots of cubic equations. Because the formula involves $i = \sqrt{-1}$, you will have to change the mode from real to complex.

Press (MODE), and select "$a + bi$" (as shown in the display).




*Note:* This will not affect ordinary calculations.






PROGRAM : CUBICEQN

: ClrHome
: Disp "A =”
: Input A
: Disp "B =”
: Input B
: Disp "C =”
: Input C
: Disp "D =”
: Input D
: $\left(C/A-B^2/\left(3A^2\right)\right)\rightarrow P$
: $\left(2B\text{^}3/(27A\text{^}3)-BC/\left(3A^2\right)+D/A\right)\rightarrow Q$
: $\sqrt[3]{}\left(-Q/2+\sqrt{}\left(Q^2/4+P\text{^}3/27\right)\right)\rightarrow U$
: $\sqrt[3]{}\left(-Q/2-\sqrt{}\left(Q^2/4+P\text{^}3/27\right)\right)\rightarrow V$
: $((U+V)-B/(3A))\rightarrow X$
: $\left(-(U+V)/2+((U-V)/2)\sqrt{}(-1)\sqrt{}(3)-B/(3A)\right)\rightarrow Y$
: $\left(-(U+V)/2-((U-V)/2)\sqrt{}(-1)\sqrt{}(3)-B/(3A)\right)\rightarrow Z$
: Disp "X1 =”
: Disp X
: Disp "X2 =”
: Disp Y
: Disp "X3 =”
: Disp Z
: Pause

PRGM [ Select the I/O Menu. ] 8 ENTER

Press ENTER to move to the second program line.

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ A ] 2nd [ TEST ] 1 ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 1 ALPHA [ A ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ B ] 2nd [ TEST ] 1 ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 1 ALPHA [ B ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ C ] 2nd [ TEST ] 1 ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 1 ALPHA [ C ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ D ] 2nd [ TEST ] 1 ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 1 ALPHA [ D ] ENTER

( ALPHA [ C ] ÷ ALPHA [ A ] − ALPHA [ B ] $x^2$ ÷ ( 3 ALPHA [ A ] $x^2$ ) ) STO▶ ALPHA [ P ] ENTER

( 2 ALPHA [ B ] ^ 3 + ( 2 7 ALPHA [ A ] ^ 3 ) −
ALPHA [ B ] ALPHA [ C ] + ( 3 ALPHA [ A ] $x^2$ ) + ALPHA
[ D ] + ALPHA [ A ] ) STO▶ ALPHA [ Q ] ENTER

MATH 4 (−) ALPHA [ Q ] + 2 + 2nd [$\sqrt{}$] ALPHA [ Q ] $x^2$ +
4 + [ P ] ^ 3 + 2 7 ) ) STO▶ ALPHA [ U ] ENTER

MATH 4 (−) ALPHA [ Q ] + 2 − 2nd [$\sqrt{}$] ALPHA [ Q ] $x^2$ +
4 + ALPHA [ P ] ^ 3 + 2 7 ) ) STO▶ ALPHA [ V ]
ENTER

( ( ALPHA [ U ] + ALPHA [ V ] ) − ALPHA [ B ] + ( 3 ALPHA
[ A ] ) ) STO▶ ALPHA [ X ] ENTER

( (−) ( ALPHA [ U ] + ALPHA [ V ] ) + 2 + ( ( ALPHA
[ U ] − ALPHA [ V ] ) + 2 ) 2nd [$\sqrt{}$] (−) 1 ) 2nd
[$\sqrt{}$] 3 ) − ALPHA [ B ] + ( 3 ALPHA [ A ] ) ) STO▶
ALPHA [ Y ] ENTER

( (−) ( ALPHA [ U ] + ALPHA [ V ] ) + 2 − ( ( ALPHA
[ U ] − ALPHA [ V ] ) + 2 ) 2nd [$\sqrt{}$] (−) 1 ) 2nd
[$\sqrt{}$] 3 ) − ALPHA [ B ] + ( 3 ALPHA [ A ] ) ) STO▶
ALPHA [ Z ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ X ] 1 2nd [ TEST ] 1
ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ X ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ X ] 2 2nd [ TEST ] 1 ALPHA [ " ] ENTER

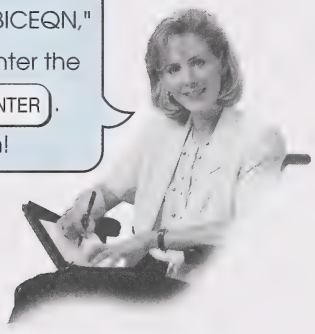
PRGM [ Select the I/O menu. ] 3 ALPHA [ Y ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ " ] ALPHA [ X ] 3 2nd [ TEST ] 1 ALPHA [ " ] ENTER

PRGM [ Select the I/O menu. ] 3 ALPHA [ Z ] ENTER

PRGM 8 ENTER

To run your program, press PRGM, select "CUBICEQN," and press ENTER. As you are prompted to enter the coefficients, enter each value and press ENTER. You are now ready to try your program!



Solve each equation.

**1.** $x^3 - 6x^2 + 11x - 6 = 0$

**2.** $2x^3 - 5x^2 + 6x - 2 = 0$

**3.** $3x^3 - x^2 - 12x + 4 = 0$

**Note:** If two of the roots of the cubic equation are non-real, they will be displayed in standard form, $a + bi$.

Compare your responses with those in the Appendix, Section 1: Follow-up Activities, Enrichment.

## Conclusion

In this section, you investigated polynomial functions. You used the Remainder Theorem to evaluate polynomials and the Factor Theorem to determine the factors of polynomials. You then solved polynomial equations and inequalities, explored problems that contain polynomials, and graphed polynomial functions using a combination of technology and analytical skills.

Your study of polynomial functions drew on the concepts and skills you developed in Pure Mathematics 10 involving linear and quadratic functions.

The study of mathematics is like a building under construction. There is always something new to add. The additions must be built on solid knowledge and understanding. Make sure you review what you have learned on a regular basis so that you are always working from a solid foundation.

## Assignment

Now, turn to Assignment Booklet 4A to complete the assignment for Section 1.

# Section 2

# The Algebra of Functions

Seen in its natural habitat, the arctic wolf is a magnificent animal. A subspecies of the grey wolf, its range is the tundra, the rolling hills and glacial valleys of the North. The arctic wolf's survival depends on a complex interplay of elements within what many would consider an unforgiving environment. Whether the wolf goes hungry or is well fed depends on the numbers of its prey, which include the arctic hare, lemmings, and caribou. These numbers, in turn, depend on the growth of vegetation during a brief arctic summer.

Mathematics, too, is a complex interplay of elements. One fundamental element is the function.

In this section, you will explore the algebra of functions. You will investigate how functions can be combined or modified to form new functions and how these new functions are related to the original functions. Throughout your study, you will use your graphing calculator to help visualize these operations and generalize the results.

# Activity 1: Operations with Functions

What strategy do you use when putting together a jigsaw puzzle? Do you first turn all the pieces so they are facing up and then sort them by colour and pattern? Do you start with the edge pieces, complete the border, and then fill in the picture section by section?

Mathematicians use functions like pieces of a puzzle. They fit them together to see the big picture—to model complex situations that cannot be represented by simpler forms. The functions they select to link together are chosen because they can represent specific parts of the problem.

In this activity, you will combine functions using the basic operations of addition, subtraction, multiplication, and division. You will then discover how to use combinations of functions to solve real-world problems.

Turn to page 242 of *MATHPOWER™ 11* and read "Operations With Functions."

**1.** Answer the following questions on page 242 of the textbook:

**a.** "Explore: Complete the Table"
**b.** questions 1 to 7 of "Inquire"

---

Compare your responses with those in the Appendix, Section 2: Activity 1.

---

**Mathematical Process**

- Communication
- [X] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

If $f$ and $g$ are functions, then their sum, difference, product, and quotient are defined as follows:

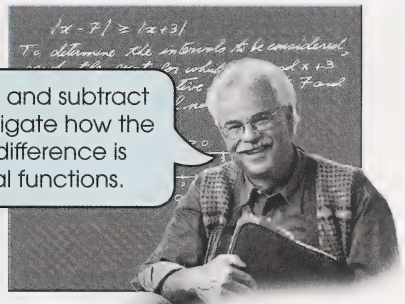
- $(f+g)(x)=f(g)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(fg)(x)=f(x)\times g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

You will now apply these definitions to specific functions.

Turn to page 243 of *MATHPOWER™ 11* and read from the top of the page to the red line on page 245, working through Examples 1 to 4.

**2.** Answer questions 1, 7, 10, 27, 29, and 35 of "Practice" on page 247 of the textbook.

Compare your responses with those in the Appendix, Section 2: Activity 1.



Now that you can add and subtract functions, you will investigate how the graph of the sum or difference is related to the original functions.

## Example

Given $f(x)=\sqrt{x-1}$ and $g(x)=x$, graph the following on the same set of axes. Describe how the graphs of the functions are related and how the domains are related.

**a.** $y=f(x)$, $y=g(x)$, and $y=(f+g)(x)$

**b.** $y=f(x)$, $y=g(x)$, and $y=(f-g)(x)$

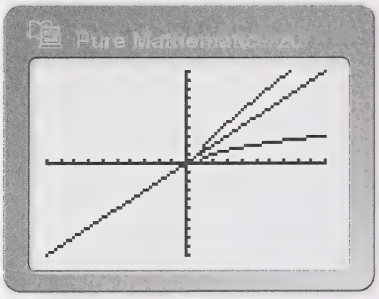
**Mathematical Process**

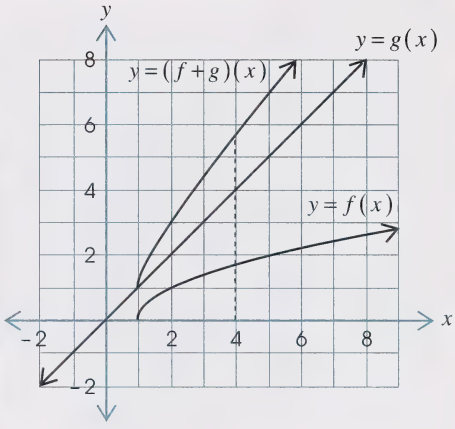
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

## Solution

**a.** $(f+g)(x) = f(g)+g(x)$

$= \sqrt{x-1}+x$






The graph of $y=(f+g)(x)$ can be obtained by adding the $y$-coordinates of the original functions for each value of $x$. If both $y$-values are positive, then it is just like taking two rods representing the $y$-coordinates of the original functions and laying them end to end to locate the $y$-coordinate of the sum.

The domain of $f(x)=\sqrt{x-1}$ is $x \geq 1$ because the square roots of negative values are non-real. Even though the domain of $g(x)=x$ is the set of real numbers, the domain of $y=(f+g)(x)$ is also $x \geq 1$.

The rule for finding the domain of a sum function is to determine the intersection of the domains of the component functions. The intersection of the domains is the "overlap" of the intervals along the $x$-axis where the component functions are defined.

This can be readily seen by establishing a table of values.

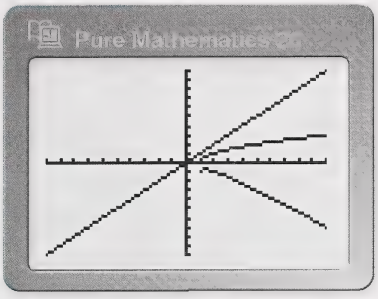
| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f(x)$ | undefined | undefined | $0$ | $1$ | $\sqrt{2}$ |
| $g(x)$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $(f+g)(x)$ | undefined | undefined | $1$ | $3$ | $3+\sqrt{2}$ |

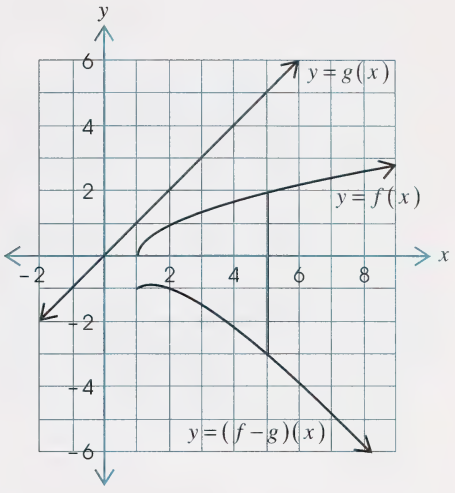
$\therefore \{x \mid x \geq 1\} \cap \{\text{Reals}\} = \{x \mid x \geq 1\}$

The symbol $\cap$ means *intersect*.

**b.** $(f-g)(x) = f(x) - g(x)$

$= \sqrt{x-1} - x$

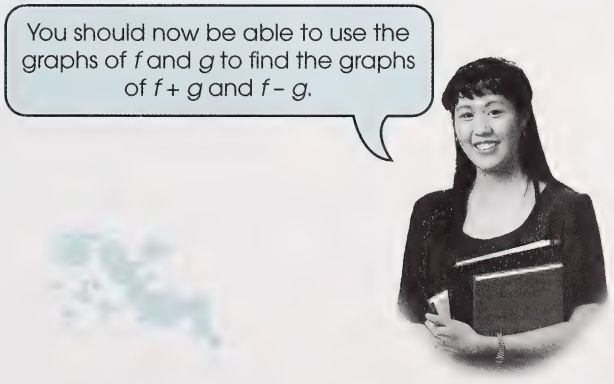



Finding the difference between two functions is similar to finding the sum, except that the $y$-coordinates of the original functions are subtracted. As for the sum, the domain of $(f-g)$ = domain of $f \cap$ domain of $g$. Thus, the domain of the difference is $x \geq 1$.

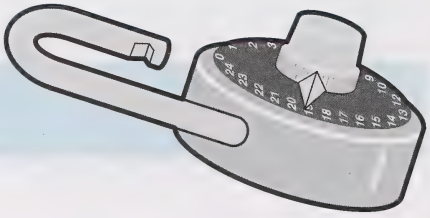
Study the following table of values.

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f(x)$ | undefined | undefined | $0$ | $1$ | $\sqrt{2}$ |
| $g(x)$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $(f+g)(x)$ | undefined | undefined | $-1$ | $-1$ | $\sqrt{2}-3$ |



**3.** Turn to page 247 of *MATHPOWER™ 11* and answer questions 15 and 17 of "Practice."

Compare your responses with those in the Appendix, Section 2: Activity 1.



Combinations of functions may be used to model real-world situations.

## Example

Air temperature usually decreases uniformly with an increase in altitude, $h$ (in metres), until the tropopause (10 000 m) is reached. This is one reason why you see snow on the mountain tops in the summer. The drop in temperature is approximately 0.6°C every 300 m.

Suppose the temperature at sea level is 20°C. Write a function, $f$, that describes the temperature at sea level. Determine a function, $g$, that describes the change in air temperature with altitude below the tropopause. Combine the previous two functions to obtain a function, $F$, that could be used to predict the air temperature at any altitude below the tropopause. Then, graph the functions.

### Solution

Let $h$ be the height (in metres) above sea level; let $f(h)$ be the temperature (in °C) at sea level; let $g(h)$ describe change in temperature (in °C); and let $F(h)$ be the air temperature (in °C) at height $h$.

Since the temperature at sea level is constant (independent of altitude), $f(h) = 20$.

Since the drop in temperature is uniform, it can be represented in the form $g(h) = kh$.

**Mathematical Process**

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Because $g(300) = -0.6$,

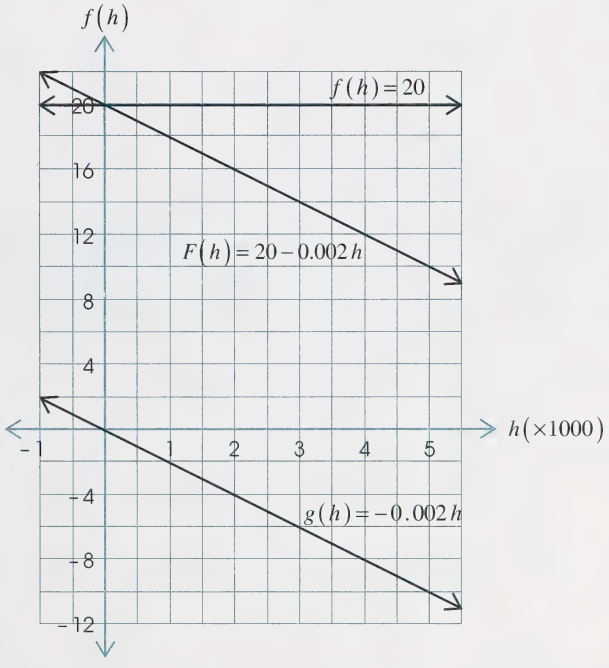
$$-0.6 = k(300)$$
$$k = -0.002$$

Therefore, $g(h) = -0.002h$, where $h \leq 10\,000$.

The air temperature at a particular altitude, $h$, is the temperature at sea level plus the temperature change for that altitude.

$$\therefore F(h) = f(h) + g(h)$$
$$= 20 - 0.002h, \text{ where } h \leq 10\,000$$

The graphs of the functions are as follows:




**4.** Use the information in the preceding example to determine the altitude at which the temperature is 11°C.

Compare your response with the one in the Appendix, Section 2: Activity 1.

Now, you will explore additional applications involving combinations of functions.

Turn to pages 245 to 247 of *MATHPOWER™ 11* and work through Examples 5 to 8.

**5.** Answer the following questions on pages 248 to 250 of the textbook:

**a.** questions 36, 39, 40, 43, 44.a., 46.a., and 53 of "Practice"
**b.** questions 54, 57, 61, 67.a., 69, and 75 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 1.

## Now Try This

Combining functions to model problem situations involves searching for patterns and testing values until the combination fits. Test these skills in the following problem.

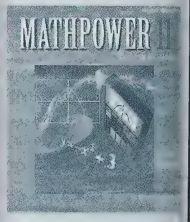
**6.** Turn to page 250 of *MATHPOWER™ 11* and answer "NUMBER POWER."

Compare your response with the one in the Appendix, Section 2: Activity 1.

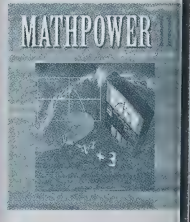
## Looking Back

In this activity, you combined functions by using the basic operations of addition, subtraction, multiplication, and division.

Use your research skills to find out what people mean when they talk about the "basics" in mathematics. Is there an agreement on what the "basics" are? Throughout your education, do you think mathematics should focus on the "basics"? Write your thoughts in your journal.




**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- Reasoning
- Technology
- Visualization




**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

# Activity 2: Composition of Functions

**Mathematical Process**
- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Sarah is thinking about buying a rocking chair that is on sale for 20% off its list price. The purchase price, $P$, is a function of the list price, $L$.

$$\begin{aligned} P &= L - 20\% \text{ of } L \\ &= L - 0.20L \\ &= 0.80L \end{aligned}$$

If Sarah decides to purchase the chair, she will have to pay GST on the purchase price. The GST is a function of the purchase price.

$$GST = 0.07P$$

The two functions can be combined so that Sarah can calculate the GST directly from the list price.

$$\begin{aligned} GST &= 0.07P \\ &= 0.07(0.80L) \\ &= 0.056L \end{aligned}$$

For example, if the rocking chair is listed at \$100, Sarah would pay \$80 after the 20% discount. The GST on \$80 is $0.07 \times \$80 = \$5.60$.

The function $GST = 0.056L$ gives the same amount.

$$\begin{aligned} GST &= 0.056L, \text{ where } L = \$100 \\ &= 0.056(100) \\ &= \$5.60 \end{aligned}$$

The process of combining functions in this fashion is called composition. Formally, the **composition of $f$ with $g$**, written $f \circ g$ or $y = f(g(x))$, is the function obtained by replacing $x$ with $g(x)$ in $y = f(x)$. In this activity, you will explore the composition of functions and its applications.

composition of $f$ with $g$: the function obtained by replacing $x$ with $g(x)$ in $y = f(x)$

1. Turn to page 251 of *MATHPOWER™ 11* and answer the following:

   **a.** questions a. to c. of "Explore: Use the Formulas"
   **b.** questions 1 to 5 of "Inquire"

Compare your responses with those in the Appendix, Section 2: Activity 2.

The composition of a function $f$ with function $g$, written $(f \circ g)(x) = f(g(x))$, must not be confused with the multiplication of the same two functions. Recall that the product is written as follows:

*Note: The small circle ($\circ$) is a composition symbol, not a multiplication symbol!*

$$(fg)(x) = f(x) \times g(x)$$

Turn to page 251 of *MATHPOWER™ 11* and read from the red line to the bottom of page 252, working through Examples 1 and 2.

2. Answer the following questions on page 256 of the textbook:

   **a.** questions 1 to 10 of "Practice"
   **b.** questions 12, 14, 16, and 18 of "Practice"

Compare your responses with those in the Appendix, Section 2: Activity 2.

In the preceding questions, you evaluated composition functions for given values in their domains. Now, you will write the general form of a composition function. Here, you will need your graphing calculator to check your work.

## Example

If $f(x) = x^2 - 8$ and $g(x) = x - 3$, determine $y = f(g(x))$. Use your graphing calculator to check your answer.

**Mathematical Process**
- [x] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

**Mathematical Process**
- [ ] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [x] Technology
- [ ] Visualization

## Solution

$$
\begin{aligned}
y &= f(g(x)) \\
&= f(x-3) && \longleftarrow g(x) = x-3 \\
&= (x-3)^2 - 8 && \longleftarrow \text{Replace } x \text{ with } x-3 \text{ in } f(x) = x^2 - 8. \\
&= (x^2 - 6x + 9) - 8 \\
&= x^2 - 6x + 1
\end{aligned}
$$

The composition function is $y = x^2 - 6x + 1$.

Use the following steps to check your answer on the graphing calculator:

**Step 1:** Enter $g(x) = x - 3$ as function $Y_1$.

[Y=] [X,T,Θ,n] [−] [3] [ENTER]

**Step 2:** Enter $f(x) = x^2 - 8$ as function $Y_2$.

[X,T,Θ,n] [$x^2$] [−] [8] [ENTER]

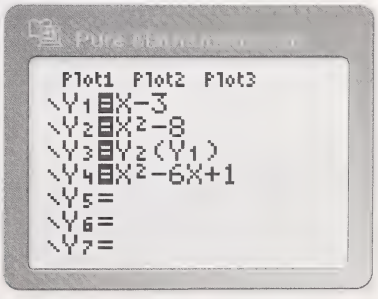
**Step 3:** Enter the composition of $f$ with $g$ as function $Y_3$.

[VARS] [ Select the Y-VARS menu. ] [1] (1: Function) [2] (2: $Y_2$) [(]
[VARS] [ Select the Y-VARS menu. ] [1] (1: Function) [1] (1: $Y_1$) [)]
[ENTER]

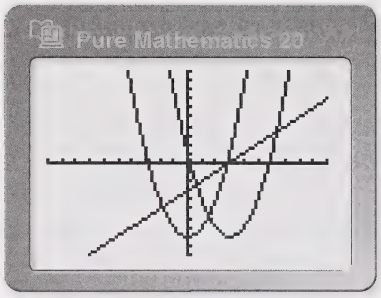
**Step 4:** Enter your answer to $y = f(g(x))$ as $Y_4$.

[X,T,Θ,n] [$x^2$] [−] [6] [X,T,Θ,n]
[+] [1]

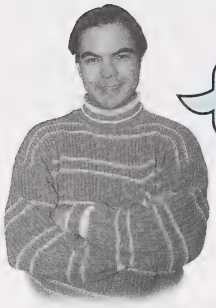
*Note: This display shows Steps 1 to 4.*



**Step 5:** Graph the functions by pressing GRAPH.




Three distinct curves appear. The calculator first graphs $g(x) = x - 3$; then it graphs $f(x) = x^2 - 8$; and, finally, it graphs the composition function $Y_2(Y_1)$. The calculator also graphs $Y_4$; but because the answer to $y = f(g(x))$ is correct, it graphs $y = x^2 - 6x + 1$ on top of the graph of $Y_2(Y_1)$. If your answer was wrong, a fourth curve would appear on the display!



Now, it is your turn to determine the general form of composition functions.

Turn to pages 253 to 255 of *MATHPOWER™ 11* and work through Examples 3 to 8.

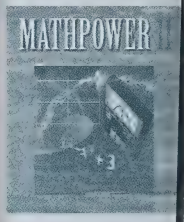
**3.** Answer the following questions on pages 256 to 259 of the textbook:

**a.** questions 22, 31, 39, 50, 51, 61, 64, and 68 of "Practice"
**b.** questions 69, 72, 73, 88, 91, and 92 of "Applications and Problem Solving"

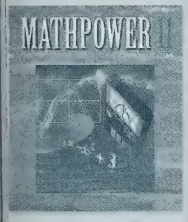
Compare your responses with those in the Appendix, Section 2: Activity 2.

From the preceding examples and questions, you discovered that the properties of composing functions are not the same as the properties of multiplying functions. For the composition of functions, order matters; for the multiplication of functions, interchanging the functions does not matter because the result does not change.

**Remember:** For most functions, $f \circ g \neq g \circ f$. However, for all functions, $fg = gf$.



**Mathematical Process**
- Communication
- Connection
- Estimation
- [x] Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization




## Now Try This

When solving a numerical problem, even when using a calculator, it is useful to look at your answer and ask yourself if it seems reasonable in the context of the problem or in light of the numbers involved in the calculation. Being able to calculate mentally is part of developing this "number sense."

In the questions that follow, you will use compatible numbers to find quotients mentally, thus adding another mental-calculation skill to your repertoire.

**4.** Turn to page 241 of *MATHPOWER™ 11* and answer questions 1 to 12 of "Mental Math: Dividing Using Compatible Numbers."

Compare your responses with those in the Appendix, Section 2: Activity 2.

**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

## Looking Back

In this activity, you examined the composition of functions. You determined both numerical values and general forms for composite functions. For the composition of two functions, you discovered that the domain of the function that is applied second is a subset of the range of the function that is applied first. You also observed that order matters, unlike the product of two functions.

In your journal, make a list of how composition and multiplication are similar. Then, make a list of how composition and multiplication are different.

# Activity 3: Inverse Functions

Most people are struck by the beauty of reflections in a calm lake. Often, in the evening, the colours are accentuated, enhancing the beauty of the reflection. Photographers try to capture the moment on film, but surely you will agree that it is better to witness the scene firsthand.

What do inverted images have to do with mathematical functions? In this activity, you will examine functions and their inverses. You will discover that the graphs of inverses are mirror images of the graphs of the original functions.

Turn to page 261 of *MATHPOWER™ 11* and read "Inverse Functions."

**1.** Answer the following questions on page 261 of the textbook:

**a.** "Explore: Complete the Table"
**b.** questions 1 to 5 of "Inquire"

---

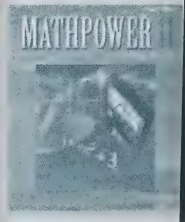
Compare your responses with those in the Appendix, Section 2: Activity 3.

---

You have just discovered that the **inverse of a function** is a relation formed by interchanging the coordinates within each ordered pair of the original function. Remember, a function is a special kind of relation—a relation in which no two ordered pairs have the same first component. Therefore, the inverse of a function could also be classified as an **inverse of a relation**.

**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization




*inverse of a function*
*a relation formed by interchanging the coordinates within each ordered pair of the original function*
*inverse of a relation*
*a relation formed by interchanging the coordinates within each ordered pair of the original relation*

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- [x] Visualization

## Example

Suppose $f = \{(-1,2), (1,3), \text{and } (4,4)\}$.

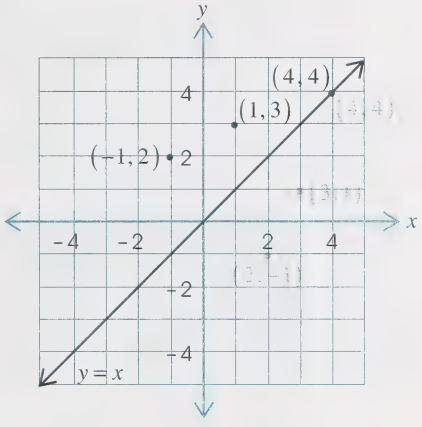
**a.** Find the inverse of function $f$.

**b.** Graph $f$ and its inverse. How are the graphs related with respect to the line $y = x$?

## Solution

**a.** Switch the first and second coordinates of each ordered pair. Thus, the inverse of $f = \{(2,-1), (3,1), \text{and } (4,4)\}$.

**b.**




*Note: Throughout this module, the graphs of inverses will be grey.*

The graph of function $f$ and the graph of its inverse appear to be mirror images of each other in the line $y = x$. The fact that $y = x$ acts as a mirror can be proven for any ordered pair and its inverse.

Consider $(1,3)$ and $(3,1)$. Join these points. The midpoint of the segment is $(2,2)$, which lies on $y = x$. Also, the slope of this segment is $-1$ and the slope of $y = x$ is 1. Therefore, the line $y = x$ intersects the segment at a right angle, thus proving that $(1,3)$ is the reflection of $(3,1)$ and vice versa.

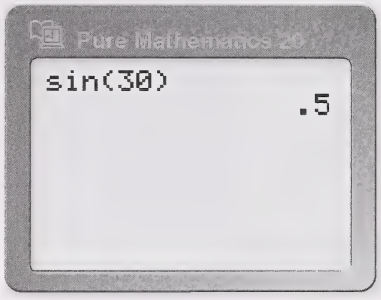
**2.** Use $P(a, b)$ and $Q(b, a)$ to prove that $y = x$ is the perpendicular bisector of segment $PQ$.

Compare your response with the one in the Appendix, Section 2: Activity 3.

In the preceding example, both $f$ and its inverse are functions. No two ordered pairs in $f$ (and likewise in its inverse) have the same first coordinate. If the inverse of a function, $f$, is also a function, the inverse function can be represented by the symbol $f^{-1}$.

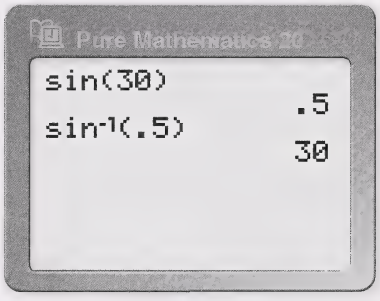
In Pure Mathematics 10, you used $-1$ to represent inverses when you studied trigonometry. For example, if you are asked to evaluate $\sin 30°$ using your graphing calculator, you would get the following:

sin 3 0 ) ENTER




Now, you are asked to find the acute $\angle\theta$ for which $\sin\theta = 0.5$. Here is where you need to use the inverse of sine $\left(\sin^{-1}\right)$ to find $\angle\theta$.

2nd [sin$^{-1}$] 0 • 5 ) ENTER




The inverse function of sine is $\sin^{-1}$. Be careful! This is an inverse function, not a reciprocal; so, $\sin^{-1}\theta \neq \frac{1}{\sin\theta}$.

To explore inverses further, turn to page 262 of *MATHPOWER™ 11* and read from the top of the page to the red line, working through Example 1.

**3.** Answer questions 1 to 4 of "Practice" on page 268 of the textbook.

Compare your responses with those in the Appendix, Section 2: Activity 3.

**Mathematical Process**

- [x] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [x] Technology
- [ ] Visualization

You have found inverses by switching coordinates in ordered pairs. How would you find the inverse of a function from its equation?

Turn to page 262 of *MATHPOWER™ 11* and read from the red line to the bottom of page 265, working through Examples 2 through 5.

**4.** Answer questions 11, 21, 26, 30, 31, 37, 45, and 47 of "Practice" on page 268 of the textbook.

Compare your responses with those in the Appendix, Section 2: Activity 3.

You have seen that the inverse of a function is not necessarily a function. If the inverse is not a function, is there a way of altering the original function so that the inverse is a function?

## Example

Graph $f(x) = x^2 + 3$ and its inverse.

### Solution

Replace $f(x)$ with $y$ for ease in finding the inverse.

$$y = x^2 + 3$$

Find the inverse.

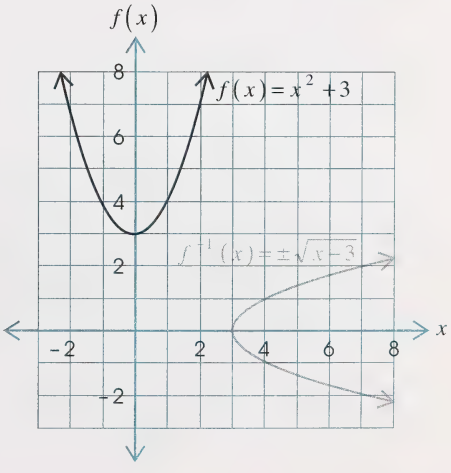
$$x = y^2 + 3$$

$$x - 3 = y^2$$

$$\pm\sqrt{x-3} = y$$

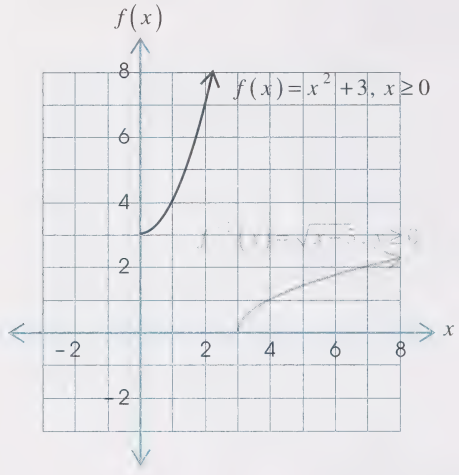
$$y = \pm\sqrt{x-3}$$

$$\therefore f^{-1}(x) = \pm\sqrt{x-3}$$



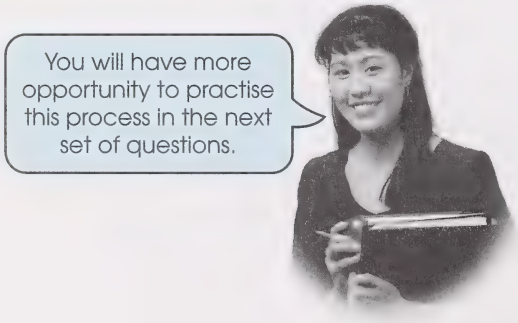
Notice from the graph that the relation $x = y^2 + 3$ is not a function, since $y$ is not uniquely determined for $x > 3$.

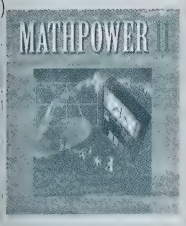
If you were to restrict the original function to $y = x^2 + 3$, where $x \geq 0$, then its inverse would be $x = y^2 + 3$, where $y \geq 0$. The negative portions of each graph would be eliminated. As a result, the inverse could be rewritten as $y = \sqrt{x-3}$ or $f^{-1}(x) = \sqrt{x-3}$.




Recall that the inverse will undo what the function does. For example, the function takes 4 into 19; that is, $f(4) = 19$. The inverse will take 19 back into 4; that is, $f^{-1}(19) = 4$.

There are many ways of restricting the domain of a function so that its inverse is also a function. In the previous example, you could have written $f(x) = x^2 + 3$, where $x \leq 0$. The inverse would be $x = y^2 + 3$, where $y \leq 0$. When you solve for $y$, the inverse is $y = -\sqrt{x-3}$ or $f^{-1}(x) = -\sqrt{x-3}$.






Turn to pages 266 and 267 of *MATHPOWER™ 11* and work through Examples 6 to 8.

**5.** Answer the following questions on pages 269 and 270 of your textbook:

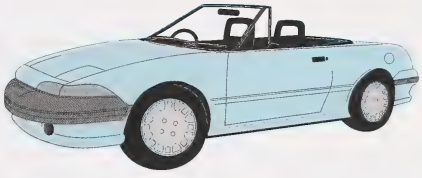
**a.** questions 68, 69, 75, 80, and 81 of "Practice"
**b.** questions 84, 86, 89, 94, and 96 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 3.

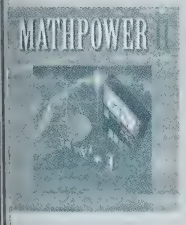
## Now Try This

**Mathematical Process**
- [x] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

When you say "is a function of" in everyday conversation, you generally mean that one thing "depends on" another. For example, suppose you are in the market for a new car. Your final decision is a function of a number of variables: cost, styling, performance, safety, and so on.



One measure of a car's performance is the time it takes to go from a complete stop to highway speeds.




Turn to page 271 of *MATHPOWER™ 11* and use the formula given in question 4 of Investigation 2, "Acceleration Times" to answer the following question:

**6.** What horsepower is needed for a 1500-kg car to accelerate from 0 km/h to 96 km/h in 8 s?

Compare your response with the one in the Appendix, Section 2: Activity 3.

## Looking Back

**Mathematical Process**
- [x] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

In this activity, you determined the inverses of functions and checked to see if the inverses were also functions. If an inverse wasn't a function, you restricted the domain of the original function so that it became a function. You then discovered that the inverse of a function undoes what the original function did. In other words, you can retrace your steps, working backwards from the answer to the original value. This is exactly what a good detective does to answer a "whodunit."

Can you think of other areas where working backwards is an essential skill? Record your thoughts in your journal.

# Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

## Extra Help

If the inverse of a function is also a function, then the function and its inverse cancel each other out: one function undoes the other. For example, if the original function maps 4 onto 10, then the inverse will map 10 onto 4. Using this line of reasoning, you should be able to find the inverse of a function by reversing the operations that the function specifies. If, for example, the function multiplies by 2, then the inverse divides by 2; if the function adds 3, then the inverse subtracts 3.

Consider the following example.

### Example

Determine the inverse of each function by reversing the operations.

**a.** $f(x)=3x-7$

**b.** $f(x)=2x^2+3$, where $x \geq 0$

### Solution

**a.** The function $f(x)=3x-7$ takes an $x$-value, multiplies it by 3, then subtracts 7 from the product.

For example, try $x=5$.

| Function, $f$ | Input | Multiply by 3 | | Subtract 7 | Output |
|---|---|---|---|---|---|
| $f(x)=3x-7$ | 5 | $\xrightarrow{\times 3}$ | 15 | $\xrightarrow{-7}$ | 8 |

$\therefore f(5)=8$

The inverse will take 8 back onto 5; that is, $f^{-1}(8)=5$.

Reverse the numbers and reverse the operations.

| Input | Add 7 | | Divide by 3 | Output | Inverse of $f$ |
|---|---|---|---|---|---|
| 8 | $\xrightarrow{+7}$ | 15 | $\xrightarrow{\div 3}$ | 5 | $f^{-1}(x)=\frac{x+7}{3}$ |

Therefore, the inverse of $f(x)=3x-7$ is $f^{-1}(x)=\frac{x+7}{3}$.

**b.** The function $f(x)=2x^2+3$, where $x \geq 0$, takes a non-negative $x$-value, squares it, multiplies the result by 2, then adds 3 to the product.

Try $x=5$.

| Function, $f$ | Input | Square | |
|---|---|---|---|
| $f(x)=2x^2+3, x \geq 0$ | 5 | $\xrightarrow{x^2}$ | 25 |

| Multiply by 2 | | Add 3 | Output |
|---|---|---|---|
| $\xrightarrow{\times 2}$ | 50 | $\xrightarrow{+3}$ | 53 |

$\therefore f(5)=53$

The inverse will take 53 onto 5; that is, $f^{-1}(53)=5$.

Reverse the numbers and reverse the operations.

| Input | Subtract 3 | | Divide by 2 | |
|---|---|---|---|---|
| 53 | $\xrightarrow{-3}$ | 50 | $\xrightarrow{\div 2}$ | 25 |

| Take the Positive Root | Output | Inverse of $f$ |
|---|---|---|
| $\xrightarrow{\sqrt{\ }}$ | 5 | $f^{-1}(x)=\sqrt{\frac{x-3}{2}}$ |

Therefore, the inverse of $f(x)=2x^2+3$, where $x \geq 0$ is $f^{-1}(x)=\sqrt{\frac{x-3}{2}}$.

Find the inverse of each function using the method described in the preceding example.

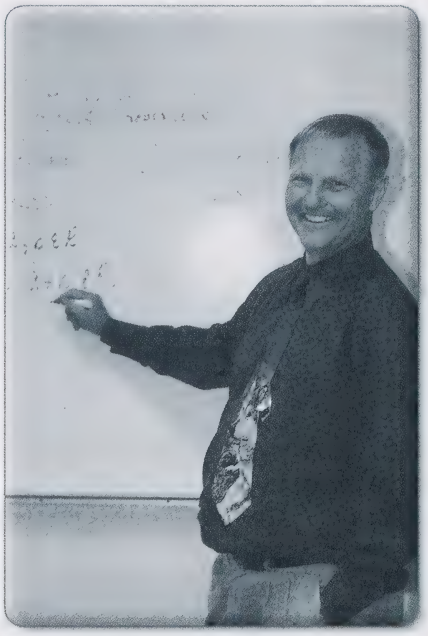
**1.** $f(x) = 3x + 4$

**2.** $f(x) = \frac{x^3 - 7}{4}$

Compare your responses with those in the Appendix, Section 2: Follow-up Activities, Extra Help.

## Enrichment

If you pursue your study of mathematics beyond high school, you may encounter abstract algebraic systems, called fields. You are already familiar with three fields: the rational-number system, the real-number system, and the complex-number system. If the common features of these fields are generalized and then examined, the results that arise will apply to all fields regardless of how bizarre the elements of these fields may appear to a non-mathematician. So, instead of looking at every system separately, a generalized system is explored.



A field is a set of elements or "numbers" together with two operations: addition and multiplication. The elements do not have to be ordinary numbers; they could be functions, for example. The two operations share many properties in common with ordinary addition and multiplication.

### Properties of Addition

**Property 1:** Closure

If $a$ and $b$ are field elements, then the sum, $a + b$, also belongs to the field. For example, if $a$ and $b$ are represented by real numbers, 3 and 4 respectively, then their sum, 7, is also a real number.

**Property 2:** Associative Property

$$a+(b+c)=(a+b)+c$$

**Property 3:** Commutative Property

$$a+b=b+a$$

**Property 4:** Zero Element

The zero element, called the additive identity element, exists such that $a+0=0+a=a$, for any field element $a$.

**Property 5:** Additive Inverses

Elements called additive inverses, $-a$, exist such that $a+(-a)=(-a)+a=0$, for any field element, $a$.

### Properties of Multiplication

**Property 1:** Closure

If $a$ and $b$ are field elements, then the product, $a\times b$, also belongs to the field. For example, if $a$ and $b$ are represented by real numbers 3 and 4, then their product, 12, is also a real number.

**Property 2:** Associative Property

$$a\times(b\times c)=(a\times b)\times c$$

**Property 3:** Commutative Property

$$a\times b=b\times a$$

**Property 4:** Multiplicative Identity Elements

The multiplicative identity element, $e$, exists such that $e\times a=a\times e=a$, for any field element $a$.

For real numbers, the multiplicative identity element is 1, because $1\times a=a\times 1=a$ for any real number $a$.

**Property 5:** Multiplicative Inverses

For every non-zero field element, $a$, there exists an element $a^{-1}$, called a multiplicative inverse, such that

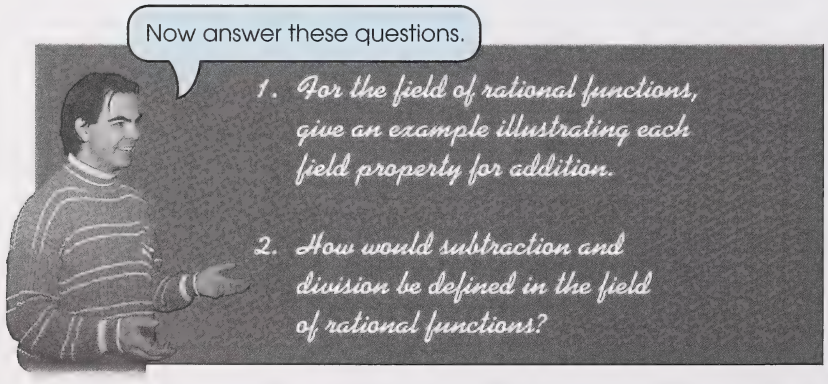
$$a\times\left(a^{-1}\right)=\left(a^{-1}\right)\times a=e$$

For a real number such as 3, the multiplicative inverse is $\frac{1}{3}$.

$$3\left(\frac{1}{3}\right)=\left(\frac{1}{3}\right)3=1$$

In addition to the preceding properties, there is a connection between multiplication and addition. Multiplication is distributive over addition; that is, $a\times(b+c)=a\times b+a\times c$.

Now, the set of rational functions together with the addition and multiplication of functions, as defined in Activity 1, constitute a field. Recall that a rational function is formed by dividing one polynomial by another. For example, $f(x)=\dfrac{x^2+x-1}{2x-1}$, $g(x)=\dfrac{2x+1}{1}$, $h(x)=3x$, and $k(x)=0$ are all rational functions.


Now answer these questions.
1. For the field of rational functions, give an example illustrating each field property for addition.
2. How would subtraction and division be defined in the field of rational functions?

Compare your responses with those in the Appendix, Section 2: Follow-up Activities, Enrichment.

## Conclusion

In this section, you added, subtracted, multiplied, and divided functions. In addition to these operations, you found the composition of functions. In each instance, you determined the domain and range of the components and of their combinations. Lastly, you determined the inverse of functions and identified your answers as functions or non-functions. Throughout this section, you applied your newly acquired skills to problem situations.

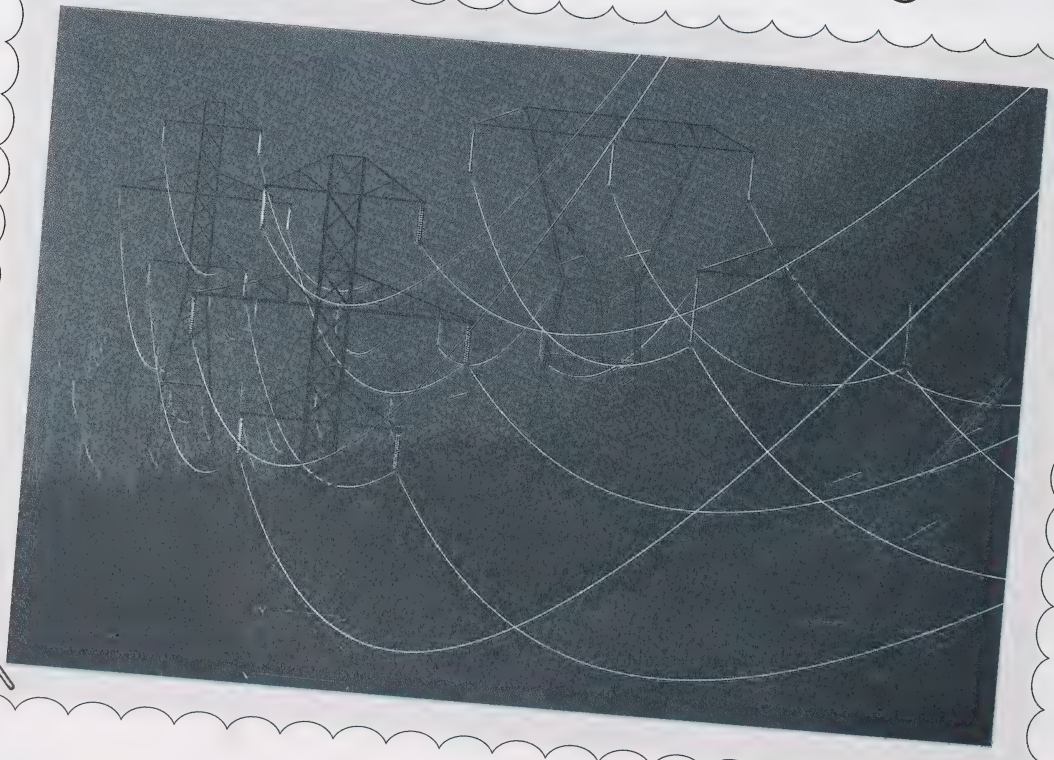
The relationships among functions can be complex, but not nearly as complex as the relationships in nature. The Canadian Arctic is a beautiful but fragile environment. To preserve its beauty for future generations, people must learn to live in harmony with the land and not threaten the delicate balance of plants, animals, and the elements.

One of the symbols of the North is the inukshuk, or man of stone. Traditionally, an inukshuk served as a marker or signpost, directing people towards game or helping them find their way around the treeless tundra. After completing this section, you should be able to find your way around the algebra of functions.

## Assignment

Now, turn to Assignment Booklet 4A to complete the assignment for Section 2.

# Section 3



# Absolute Value, Rational, and Radical Functions

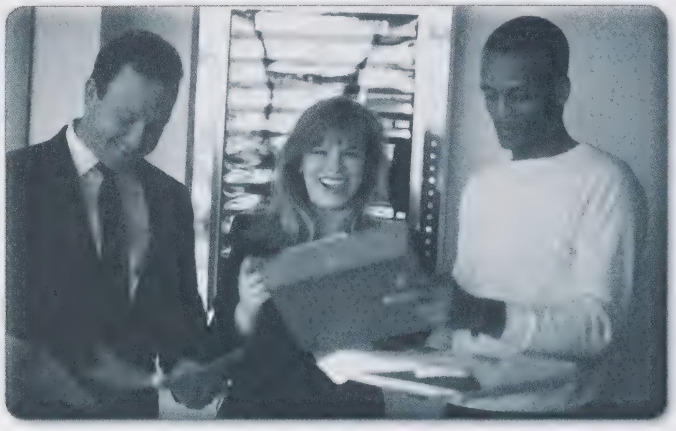
Have you heard overhead power lines (or wires) hum on a windy day? If the wind is steady and is blowing at the right speed, the power lines will vibrate like the strings of a guitar! The frequencies of the sounds produced by the resonating wires are functions of a number of variables, such as the length of the wire, the tension in the wire, and the wire's mass per unit length. The formula for these frequencies, $v_n$, is

$$v_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}}, \text{ where } n = 1, 2, 3, \ldots$$

This formula is an example of a radical function, where $L$ is the length of the wire, $F$ is the tension in the wire, and $\mu$ is the wire's mass per unit length. The lowest frequency, or fundamental frequency, is obtained by replacing $n$ with 1. Higher frequencies, called overtones, are multiples of the fundamental frequency.

In this section, you will explore several different types of functions. You will define and graph absolute value, rational, and radical functions. Using your graphing calculator and pencil and paper, you will solve equations and inequalities based on these functions. You will also solve problems and investigate real-world applications.

# Activity 1: Absolute Value Functions, Equations, and Inequalities



Have you, like the lady in the photograph, anxiously waited for a parcel and, once it arrived, were eager to open it? Suppose you telephoned a delivery company and were told that your parcel would arrive in an hour, give or take 15 minutes. The inequality that describes the time, $t$, you have to wait until you receive the parcel is $|t-60| \leq 15$, which is an inequality involving absolute value.

In this activity, you will explore functions, equations, and inequalities involving absolute value.

Do you recall the definition of **absolute value**? The absolute value of a real number is its distance from the origin on the number line.

A point at $-3$ on the number line is 3 units from the origin; a point at 2 is 2 units from the origin.

Because distance is non-negative, the absolute value of $-3$, written $|-3|$, is 3. Similarly, $|2| = 2$.

Therefore, for any real number, $x$, if $x \geq 0$, then $|x| = x$; and if $x < 0$, then $|x| = -x$.

For example, $|-3| = -(-3)$
$= 3$

**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

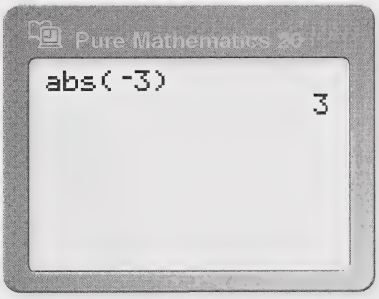
*absolute value: the distance a real number lies from the origin on the number line*

**Mathematical Process**

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

To evaluate $|-3|$ on your graphing calculator, press MATH, select the NUM menu, and press 1. This will paste "abs(" onto the screen. Next, press (–) 3 ) ENTER.

The display should look as follows:




A function with a variable within the absolute value symbol is an **absolute value function**. You will now explore absolute value functions, their graphs, and **absolute value equations**.

Turn to page 288 of *MATHPOWER™ 11* and read "Absolute Value Functions, Equations, and Inequalities."

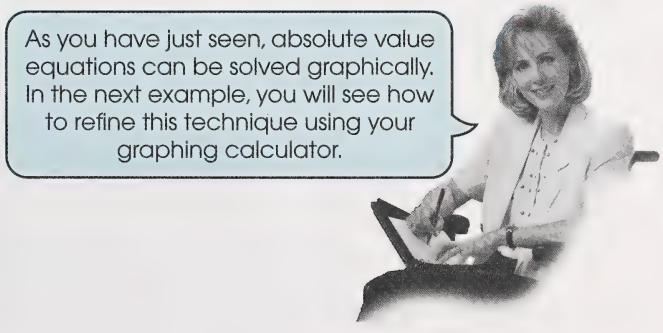
**1.** Answer the following questions on page 288 of the textbook:

**a.** questions a. and b. of "Explore: Interpret the Graphs"
**b.** questions 1 to 5 of "Inquire"

---

Compare your responses with those in the Appendix, Section 3: Activity 1.

---


As you have just seen, absolute value equations can be solved graphically. In the next example, you will see how to refine this technique using your graphing calculator.

*Absolute value function: a function with a variable within an absolute value symbol*

*Absolute value equation: an equation with a variable within an absolute value symbol*

## Example

Use your graphing calculator to solve $|2x-3|=5$.

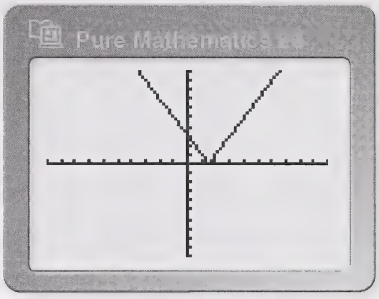
### Solution

Graph $y=|2x-3|$ and $y=5$. The roots of the equation $|2x-3|=5$ are the $x$-coordinates of the points of intersection of these two graphs.

First, using the standard window settings, graph $y=|2x-3|$.

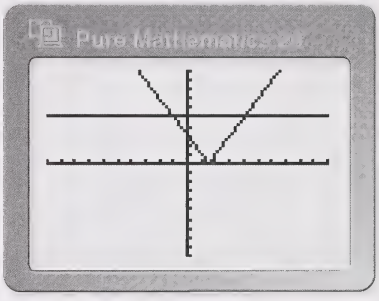
Y= MATH [ Select the NUM menu. ]

1 2 X,T,Θ,n − 3 ) GRAPH




Next, graph $y=5$. Because this will be the second curve, enter this equation at $Y_2$.

Y= 5 GRAPH

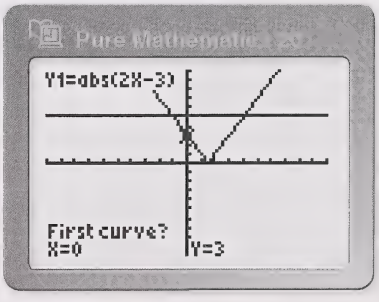



Now that you have graphed both functions, the solutions to the original equation will be the $x$-coordinates at the points of intersection.

Use the Intersect feature from the CALCULATE menu to determine where the curves cross.

Press 2nd [ CALC ] 5 (5: intersect).

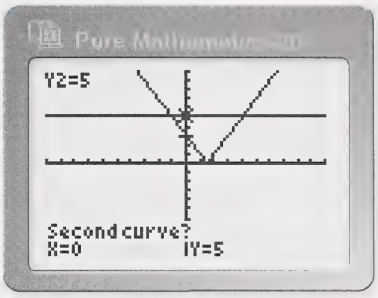
The calculator asks if you wish to select the graph of $y=|2x-3|$ as the first curve. To answer "yes," press ENTER.



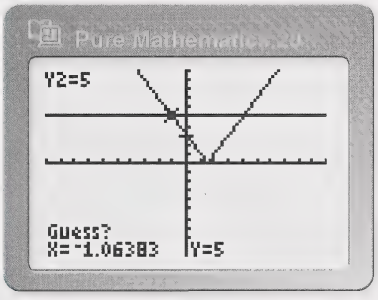

**Mathematical Process**

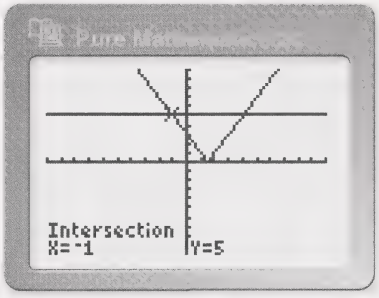
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- [x] Visualization

Now, the blinking cursor lies on the graph of $y = 5$. At this point, the calculator asks if you wish to select this graph as the second curve. To answer "yes," press ENTER.




Next, the calculator prompts you to enter your guess. Use the arrow keys to position the cursor on or close to the point of intersection on the left; then press ENTER.

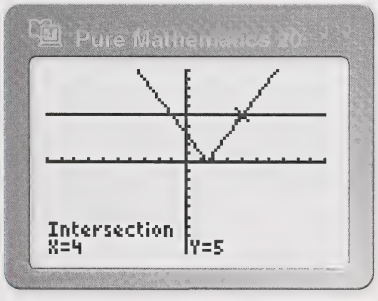



Therefore, one of the roots of the original equation is $x = -1$.

Now, find the second root using the same procedure.

2nd [ CALC ] 5 [ Position the cursor near the second point of intersection. ]
ENTER ENTER ENTER




Recall that you press ENTER the first two times to select graphs $Y_1$ and $Y_2$.

Therefore, the other root of the original equation is $x = 4$.

**Check**

Check the roots by substituting them into the original equation.

For $x = -1$,

| LS | RS |
|---|---|
| $\lvert 2x-3\rvert$ | 5 |
| $=\lvert 2(-1)-3\rvert$ | |
| $=\lvert -5\rvert$ | |
| $=5$ | |
| LS = | RS |

For $x = 4$,

| LS | RS |
|---|---|
| $\lvert 2x-3\rvert$ | 5 |
| $=\lvert 2(4)-3\rvert$ | |
| $=\lvert 5\rvert$ | |
| $=5$ | |
| LS = | RS |

The roots of $\lvert 2x-3\rvert = 5$ are $-1$ and 4.

Next, you will look at pencil-and-paper techniques for solving absolute value equations.

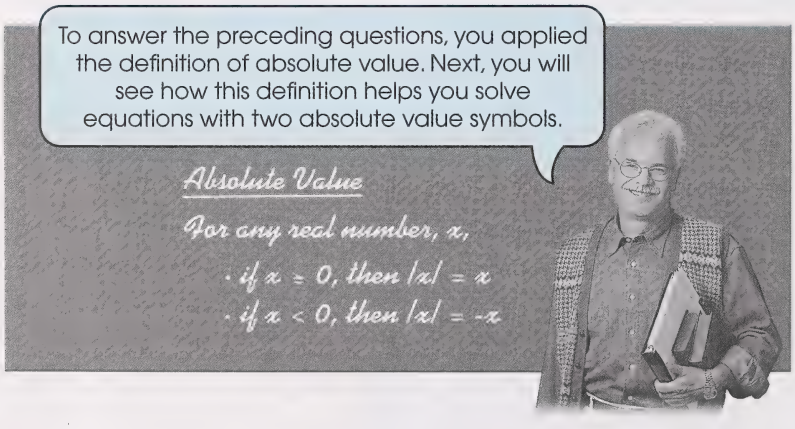
Turn to page 289 of *MATHPOWER™ 11* and read from the top of the page to the red line near the top of page 290, working through Examples 1 and 2.

**2.** Answer questions 1, 8, 12, 20, 21, and 28 of "Practice" on page 296 of the textbook.

---

Compare your responses with those in the Appendix, Section 3: Activity 1.

---


To answer the preceding questions, you applied the definition of absolute value. Next, you will see how this definition helps you solve equations with two absolute value symbols.
Absolute Value
For any real number, x,
· if x ≥ 0, then |x| = x
· if x < 0, then |x| = -x

Turn to page 290 of *MATHPOWER™ 11* and read from the red line near the top of the page to the bottom of page 291, working through Examples 3 and 4.

**3.** Answer questions 35, 38, 42, and 46 of "Practice" on page 296 of the textbook.

Compare your responses with those in the Appendix, Section 3: Activity 1.

*solute value*
*quality: an*
*quality that has a*
*riable within an*
*solute value symbol*

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

## Example

Solve each inequality, and graph your solution on a number line.

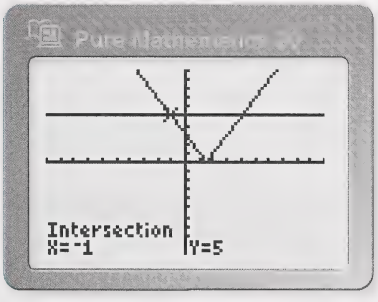
**a.** $|2x-3| \leq 5$

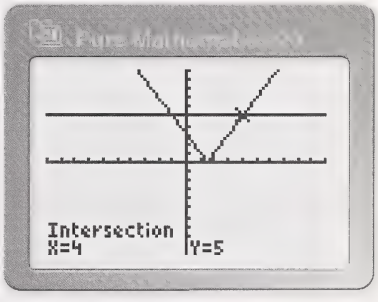
**b.** $|2x-3| > 5$

## Solution

**a. Method 1: Using a Graphing Calculator**

Graph $y = |2x-3|$ and $y = 5$ on your graphing calculator. Use the Intersect feature from the CALCULATE menu to determine where these two graphs intersect.




For $x = -1$ and $x = 4$, $|2x-3| = 5$.

The graph of $y = |2x-3|$ lies below the graph $y = 5$ when $-1 < x < 4$.

Therefore, $|2x-3| \leq 5$ when $-1 \leq x \leq 4$.

When you graph this interval on a number line, remember to include the endpoints $x = -1$ and $x = 4$ using solid dots at $-1$ and 4.

**Method 2: Using the Definition of Absolute Value**

The inequality is of the form $|\quad| \leq 5$. If you replace the rectangle with any real number between $-5$ and $+5$, the statement is true. For example, $|-4| < 5$, $|-1| < 5$, and $|3| < 5$.

$\therefore \; -5 \leq \quad \leq 5$

$-5 \leq 2x - 3 \leq 5$

Next, isolate $x$ in the statement.

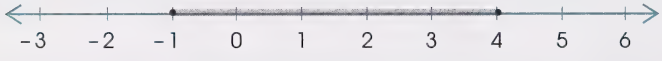
$-5 + 3 \leq 2x - 3 + 3 \leq 5 + 3$ ← Add 3 to each part.

$-2 \leq 2x \leq 8$

$\frac{-2}{2} \leq \frac{2x}{2} \leq \frac{8}{2}$ ← Divide each part by 2.

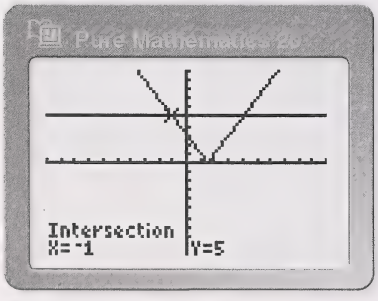
$-1 \leq x \leq 4$

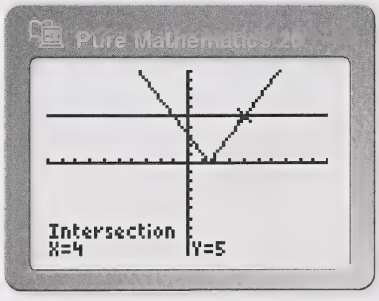
Therefore, $|2x - 3| \leq 5$ when $-1 \leq x \leq 4$.




**b. Method 1: Using a Graphing Calculator**

Graph $y = |2x - 3|$ and $y = 5$ on your graphing calculator, and determine the points of intersection.




Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

The graph of $y = |2x - 3|$ lies above the graph $y = 5$ when $x < -1$ or when $x > 4$.

Therefore, $|2x - 3| > 5$ when $x < -1$ or $x > 4$.

When you graph this interval on a number line, remember to exclude the endpoints $x = -1$ and $x = 4$ using open circles.

**Method 2: Using the Definition of Absolute Value**

The inequality is of the form $|\square| > 5$. If you replace the rectangle with any real number larger than 5 or smaller than $-5$, the statement is true.

For example, $|6| > 5$, $|7| > 5$, $|8| > 5$, ... or $|-6| > 5$, $|-7| > 5$, $|-8| > 5$, ....

$\therefore \square > 5$ or $\square < -5$

$2x - 3 > 5$ $\quad$ $2x - 3 < -5$

Next, isolate $x$ in each statement.

$2x - 3 + 3 > 5 + 3$ or $2x - 3 + 3 < -5 + 3$

$2x > 8$ $\quad$ $2x < -2$

$x > 4$ $\quad$ $x < -1$

Therefore, $|2x - 3| > 5$ when $x > -1$ or $x < 4$.

**4.** Turn to page 296 of *MATHPOWER™ 11* and answer questions 49, 50, 56, 59, and 61 of "Practice."

Compare your responses with those in the Appendix, Section 3: Activity 1.

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

The preceding questions involved a single absolute value symbol. The next set of examples and questions involve inequalities with two absolute value symbols. These inequalities can be solved algebraically or graphically.

Turn to pages 292 to 295 of *MATHPOWER™ 11* and work through Examples 6 to 10.

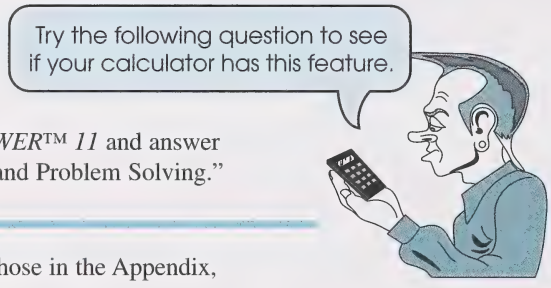
**5.** Answer the following questions on pages 296 and 297 of the textbook:

**a.** questions 63, 66, 73, 76, 83, and 91 of "Practice"
**b.** questions 99, 103, and 111.a. of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 3: Activity 1.

## Now Try This

You have used your graphing calculator to solve inequalities by analysing the graphs of systems of equations. However, some graphing calculators can be used to graph the solutions to absolute value inequalities directly.



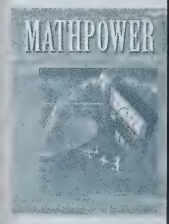

**6.** Turn to page 298 of *MATHPOWER™ 11* and answer question 112 of "Applications and Problem Solving."

Compare your responses with those in the Appendix, Section 3: Activity 1.

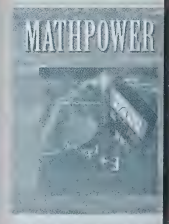
## Looking Back

In this activity, you explored absolute value functions, equations, and inequalities. You used your graphing calculator to graph absolute value functions and solve equations and inequalities. You also practised pencil-and-paper techniques based on the definition of absolute value.

In your journal, write examples of questions involving absolute value functions, equations, and inequalitites with worked solutions to help you review these topics. Share your work with another student taking Pure Mathematics 20.




**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization




**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

# Activity 2: Rational Functions, Equations, and Inequalities

Have you wondered what it would be like to travel in space beyond the solar system to the stars? Among the problems that space travel presents are the vast distances to even the nearby stars in the Milky Way, let alone to stars in other galaxies. But, how do astronomers determine distances to the stars? One method is to compare the intensity of the photographic image of a distant star with the intensity of a nearby star that is the same size and type. The intensity, $y$, of starlight that falls on a photographic plate decreases with the distance, $x$, of the star, as described by the following function:

$$y = \frac{k}{x^2}$$

For example, if the intensity of light received from one star is 100 times the intensity of light received from a second identical star, then the second star is 10 times farther away than the first star.

If you have access to the Internet, you can obtain more information on astronomy, space, spacecraft, and measuring the distances of stars at the following sites:

**http://imagine.gsfc.nasa.gov/docs/ask_astro/answers/970415c.html**
**http://www-spof.gsfc.nasa.gov/stargaze/Sintro.htm**
**http://www.nasa.gov/**

The function $y = \frac{k}{x^2}$ is an example of a **rational function**. In this activity, you will investigate rational functions and their graphs, **rational equations**, and **rational inequalities** that are based on rational functions.

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

*tional function: a ction of the form* $f(x) = \frac{g(x)}{h(x)}$, *where* $g(x)$ *and* $h(x)$ *are lynomials and* $h(x) \neq 0$

*tional equation: an ation formed by ting a rational pression equal to 0*

*tional inequality: an quality containing ational expression*

Turn to page 299 of *MATHPOWER™ 11* and read "Rational Functions, Equations, and Inequalities."

**1.** Answer the following questions on pages 299 and 300 of the textbook:

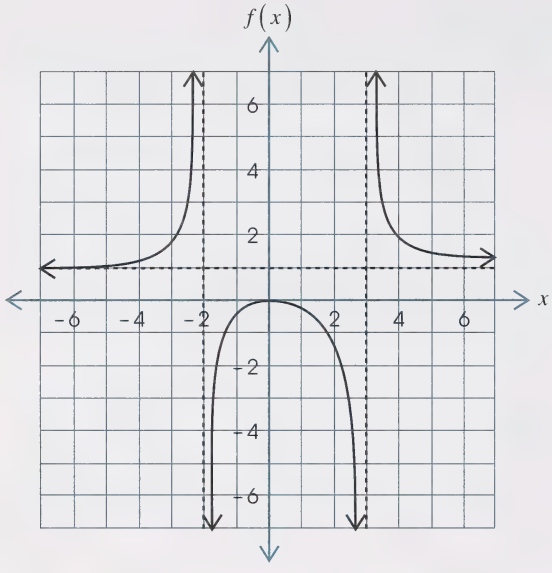
**a.** questions a. and b. of "Explore: Draw a Graph"
**b.** questions 1 to 6 of "Inquire"

---

Compare your responses with those in the Appendix, Section 3: Activity 2.

---

The graph of $f(x)=\frac{x^2}{x^2-x-6}$ is typical of a rational function.



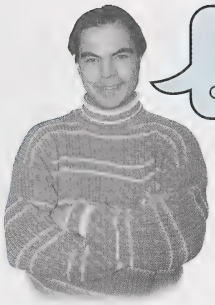

Notice that there are breaks in the graph. Unlike the graphs of polynomial functions, which produce **continuous graphs**, the graphs of some rational functions produce discontinuous graphs. The preceding graph is **discontinuous** at $x=-2$ and $x=3$.

You will also notice, as you go outward from the origin, that the graph approaches the horizontal line $f(x)=1$ and the vertical lines $x=-2$ and $x=3$. These lines are called **asymptotes**.

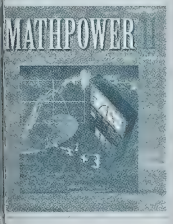
*continuous graph: graph with no breaks in the curve*

*discontinuous graph: a graph with a break in the curve*

*asymptote: a line a graph approaches more and more closely*



Now, you are going to graph rational functions, identify discontinuities, and determine horizontal and vertical asymptotes.




Turn to page 300 of *MATHPOWER™ 11* and read from the red line to the bottom of page 303, working through Examples 1 to 4.

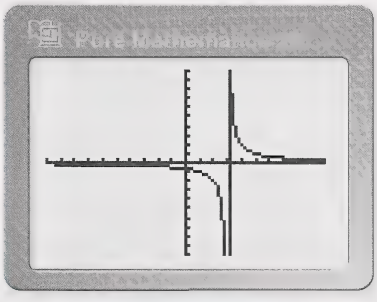
**2.** Answer the following questions on pages 308 to 310 of the textbook:

**a.** questions 1, 2, 4, 8 to 12, 15, 21, 27, 37, and 46 of "Practice"
**b.** question 97 of "Applications and Problem Solving"

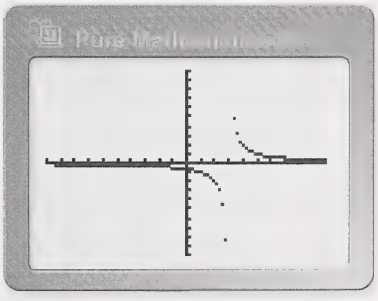
Compare your responses with those in the Appendix, Section 3: Activity 2.

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

If you graph a rational function, such as $y = \frac{2}{x-3}$, in Connected mode on your graphing calculator, there will be a vertical segment connecting the right-most pixel on the left branch of the curve to the left-most pixel on the right branch. If you do not like this feature, select "Dot" from the MODE menu to eliminate the line.



Connected Mode



Dot Mode

Using your graphing calculator, you can solve rational equations and inequalities in the same way that you solved absolute value equations and inequalities.

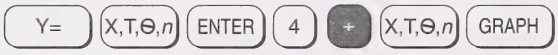
## Example

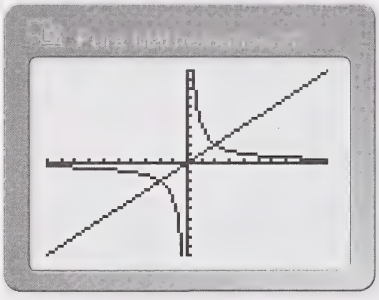
Use your graphing calculator to solve the following:

**a.** $x = \frac{4}{x}$ **b.** $x \geq \frac{4}{x}$

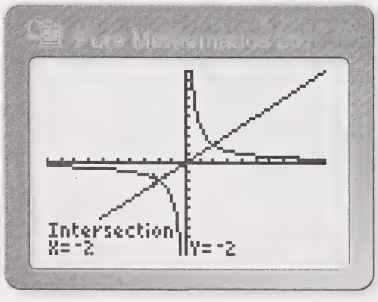
## Solution

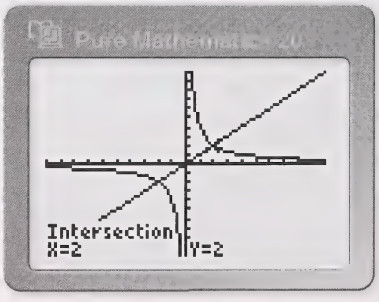
**a.** Begin by graphing the system $y = x$ and $y = \frac{4}{x}$. Make sure your calculator is in Connected mode.



The solution of the equation $x = \frac{4}{x}$ are the $x$-coordinates of the points of intersection. Use the Intersect feature from the CALCULATE menu.




**Mathematical Process**

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- [x] Visualization

If necessary, refer to Section 3, Activity 1 to refresh your memory on how to use the Intersect feature.

**Check**

Substitute $x = -2$ and $x = 2$ into the equation to check your solution.

For $x = -2$,

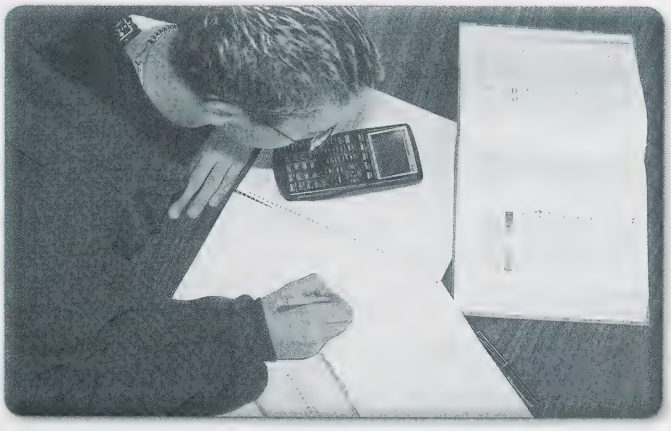
| LS | RS |
|---|---|
| $x$ | $\frac{4}{x}$ |
| $= -2$ | $= \frac{4}{-2}$ |
| | $= -2$ |

LS = RS

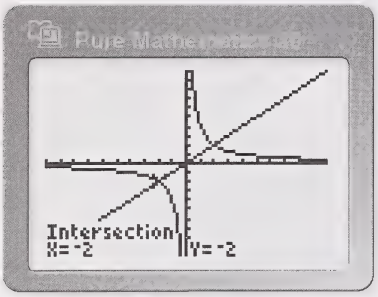
For $x = 2$,

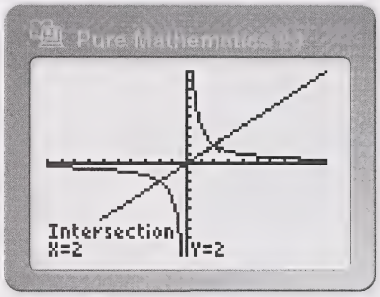
| LS | RS |
|---|---|
| $x$ | $\frac{4}{x}$ |
| $= 2$ | $= \frac{4}{2}$ |
| | $= 2$ |

LS = RS



**b.** First, graph the system $y = x$ and $y = \frac{4}{x}$ on your graphing calculator and determine the points of intersection.




To solve $x \geq \frac{4}{x}$, identify those $x$-values for which the graph of $y = x$ lies on or above the graph of $y = \frac{4}{x}$.

*Note: If point $A(x_1, y_1)$ lies above point $B(x_2, y_2)$, then $y_1 > y_2$.*

The graph of $y = x$ lies on or above the left branch of $y = \frac{4}{x}$ when $-2 \leq x < 0$. Do not include 0, because $y = \frac{4}{x}$ is not defined for that value.

The graph of $y = x$ lies on or above the right branch of $y = \frac{4}{x}$ when $x \geq 2$.

Therefore, the solution to $x \geq \frac{4}{x}$ is $-2 \leq x < 0$ or $x \geq 2$.

The solution can be graphed on a number line.

Now, you will explore both graphical and algebraic techniques for solving rational equations and inequalities. Make certain to check for **extraneous roots**.

Turn to pages 304 to 308 of *MATHPOWER™ 11* and work through Examples 5 to 8.

**3.** Answer the following questions on pages 309 to 312 of the textbook. Check your algebraic solutions using your graphing calculator.

**a.** questions 49, 54, 62, 69, 74, 77, 82, and 95 of "Practice"
**b.** questions 110, 111, 113, and 127 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 3: Activity 2.

**Mathematical Process**

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [X] Technology
- Visualization

*extraneous root: a number obtained in the process of solving an equation that does not satisfy the equation*

## Now Try This

Rational functions have many applications in the physical sciences. One of these applications involves the Doppler effect. The Doppler effect is the apparent shift in frequency of a sound source as it first approaches you and then moves away. The whistle of a train, the siren of an ambulance, or the engine of a truck sounds lower in pitch as it passes you.

**4.** Turn to page 312 of *MATHPOWER™ 11* and answer question 126 of "Applications and Problem Solving."

Compare your responses with those in the Appendix, Section 3: Activity 2.

## Looking Back

In this activity, you defined and graphed rational functions. You analysed these graphs and identified discontinuities and horizontal and vertical asymptotes. Lastly, you used graphical and algebraic techniques to solve rational equations and inequalities.

In your journal, describe how you would use your graphing calculator to graph rational functions, identify discontinuities and asymptotes, and solve rational equations and inequalities. List the features you would use and the keystrokes required to access them.

# Activity 3: Radical Functions, Equations, and Inequalities

The Andromeda Galaxy is one of the most distant objects you can see in the night sky with the naked eye. It lies about 2.3 million light-years from the Milky Way and is believed to contain twice as many stars. Even at the speed of light, travel to this galaxy is out of the question. Is it even possible for spacecraft to attain the speed of light? According to special relativity, which has been verified in laboratory experiments, the mass of an object increases without bound as it approaches the speed of light.

NOAO

The relativistic mass, $m(v)$, of an object with rest mass, $m$, is given by the following formula:

$$m(v) = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In this formula, $v$ is the speed of the object and $c$ is the speed of light. As the mass's velocity, $v$, approaches $c$, the fraction $\frac{v^2}{c^2}$ approaches 1 and the denominator of the function approaches 0. If the relativistic mass becomes infinite, the forces required to accelerate that mass become infinite. Unfortunately, space travel is limited by physical laws.

The formula given for relativistic mass is an example of a **radical function**. In this activity, you will explore radical functions, **radical equations**, and **radical inequalities**.

Turn to page 314 of *MATHPOWER™ 11* and read "Radical Functions, Equations, and Inequalities."

**1.** Answer the following questions on page 314 of the textbook:

**a.** "Explore: Draw a Graph"

**b.** questions 1 to 4 of "Inquire"

Compare your responses with those in the Appendix, Section 3: Activity 3.

**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

*radical function* a function with a variable within a radicand

*radical equation* an equation with a variable within a radicand

*radical inequality* an inequality with variable within a radicand

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

You can use your graphing calculator to graph radical functions. From these graphs, you can determine the domain and range.

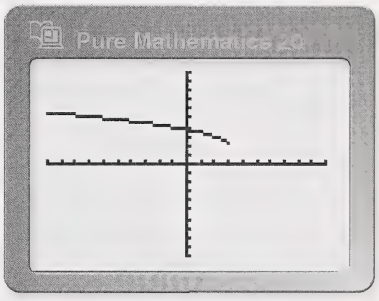
## Example

Graph $y = \sqrt{3-x} + 2$ on your graphing calculator, and determine the domain and range.

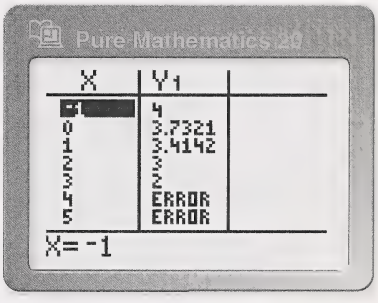
### Solution

Using the standard window settings, graph the function on your graphing calculator.

Y= 2nd [√] 3 − X,T,Θ,n ) + 2 GRAPH




The domain and range can be determined from a table of values. Press 2nd [ TABLE ].



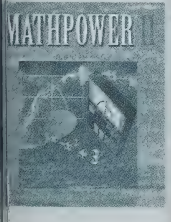

Error messages appear for those values of $x$ for which $\sqrt{3-x}$ is undefined. Therefore, the domain is $x \leq 3$, and the range is $y \geq 2$.

Now, it's your turn!

Turn to pages 314 and 315 of *MATHPOWER™ 11* and work through Example 1.

**2.** Answer questions 3, 11, and 19 of "Practice" on page 323 of the textbook.

Compare your responses with those in the Appendix, Section 3: Activity 3.



You will now practise solving radical equations.

At this point, you may wish to review the graphing calculator techniques for solving absolute value equations and rational equations outlined in Activities 1 and 2 of this section. You can use your graphing calculator to solve radical equations in exactly the same way!

The algebraic approach to solving a radical equation containing square roots involves squaring both sides of the equation.

**Mathematical Process**

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

## Example

Solve and check $\sqrt{3-x}=3$.

### Solution

$$\sqrt{3-x}=3$$

$$\left(\sqrt{3-x}\right)=3^2 \quad \longleftarrow \text{Square both sides to eliminate the root.}$$

$$3-x=9$$

$$-x=6$$

$$x=-6$$

**Check**

| LS | RS |
|---|---|
| $\sqrt{3-x}$ | 3 |
| $=\sqrt{3-(-6)}$ | |
| $=\sqrt{9}$ | |
| $=3$ | |

LS = RS

Use the positive square root only when checking your answer because the symbol $\sqrt{\ }$ indicates a principal (or non-negative) square root.

The solution is $x=-6$.

Verifying your answer is an integral part of your solution. Often, squaring both sides of an equation introduces extraneous roots. You must verify that every root you obtain for a radical equation satisfies the original equation. The following shows how squaring introduces roots.

Suppose $x = a$.

$$x^2 = a^2 \quad \leftarrow \text{Square both sides.}$$
$$x^2 - a^2 = 0 \quad \leftarrow \text{Express in standard form.}$$
$$(x-a)(x+a) = 0 \quad \leftarrow \text{Factor.}$$
$$\therefore x - a = 0 \quad \text{or} \quad x + a = 0 \quad \leftarrow \text{Zero Product Property}$$
$$x = a \qquad\qquad x = -a \quad \leftarrow \text{Solve for } x.$$

However, the original equation states that $x = a$. Therefore, $x = -a$ is an introduced or extraneous root.

**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- [x] Reasoning
- Technology
- Visualization

Turn to page 315 of *MATHPOWER™ 11* and read from the red line to the red line near the bottom of page 318, working through Examples 2 to 6.

**3.** Answer the following questions on pages 323 to 327 of the textbook:

**a.** questions 30, 31, 44, 48, 53, and 60 of "Practice"

**b.** questions 109.a., 114, 119, 125, 129, and 140.a. of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 3: Activity 3.

Like absolute value inequalities and rational inequalities, you can solve radical inequalities graphically using the same techniques. You may wish to review these techniques in Activities 1 and 2 of this section.

Like radical equations, algebraic solutions to radical inequalities containing square roots involve squaring. Once again, checking for extraneous roots is an integral part of your solution.

Turn to page 318 of *MATHPOWER™ 11* and read from the red line near the bottom of the page to the bottom of page 322, working through Examples 7 to 9.

**4.** Answer the following questions on pages 323 to 326 of the textbook:

**a.** questions 68, 73, 78, 83, and 95 of "Practice"
**b.** questions 103.a., 122, 132.a., and 132.c. of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 3: Activity 3.

## Now Try This

Solving equations and inequalities involves comparing numbers. Often, several numbers must be tested before solutions are completed. You will now try your hand at a number puzzle that involves comparing numbers and their combinations. You will likely have to try several possibilities before you succeed. Good luck!

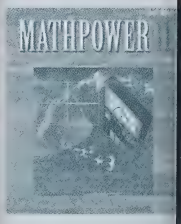
**5.** Turn to page 327 of *MATHPOWER™ 11* and answer "NUMBER POWER."

Compare your response with the one in the Appendix, Section 3: Activity 3.

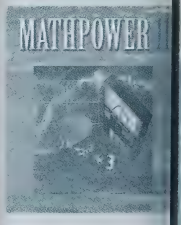
## Looking Back

In this activity, you graphed, described, and analysed radical functions with the help of your graphing calculator. In addition, you formulated and applied strategies to solve radical equations and inequalities.

There are similarities among graphical techniques and among algebraic strategies for solving absolute value, rational, and radical inequalities. Summarize these similarities in your journal. Do these techniques have a more general application? Explain.




**Mathematical Process**
- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- Reasoning
- Technology
- Visualization




**Mathematical Process**
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

# Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

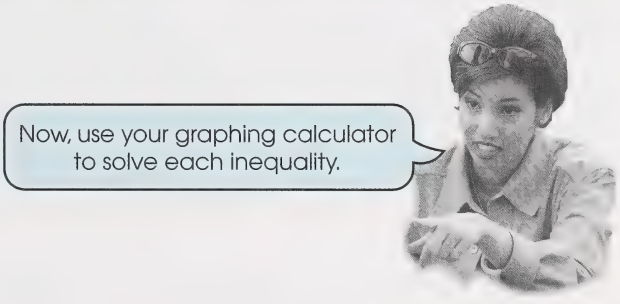
## Extra Help

Algebraic and graphing techniques used to solve equations and inequalities complement one another. Graphing calculators can be particularly useful in analysing complex functions and clarifying solutions to complicated equations and inequalities. The next set of questions provides extra practice in exploring rational functions using your graphing calculator.

Turn to "TECHNOLOGY" on page 313 of *MATHPOWER™ 11* and read "Exploring Rational Functions With a Graphing Calculator."

**1.** Answer questions 1, 3, 5, and 9 of Investigation 1, "Exploring Rational Functions" on page 313 of the textbook.

Compare your responses with those in the Appendix, Section 3: Follow-up Activities, Extra Help.

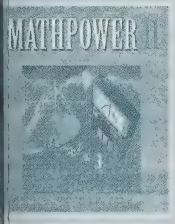



**2.** Turn to pages 330 and 331 of *MATHPOWER™ 11* and answer questions 48, 63, and 72 of "Review."

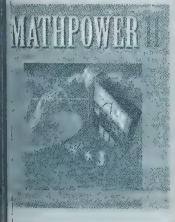
Compare your responses with those in the Appendix, Section 3: Follow-up Activities, Extra Help.



Mathematical Process
Communication
Connection
Estimation
Mental Math
Problem Solving
Reasoning
Technology
Visualization




Mathematical Process
Communication
Connection
Estimation
Mental Math
Problem Solving
Reasoning
Technology
Visualization



## Enrichment

When you see a photograph of Earth taken from space, it becomes easier to think of Earth as just another planet, conforming to the same laws of motion and gravitation. Sir Isaac Newton, born in Britain in 1642, was the first natural scientist to describe the mathematical laws of gravitation. Newton showed that all objects in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance from their centres of mass. Doubling your distance from the centre of Earth reduces the gravitational force to a quarter. This principle, known as the Universal Law of Gravitation, was just one of Newton's contributions to physics and mathematics. Viewed by many as the greatest scientist of all time, Newton laid the basis for calculus; contributed to optics; and described wave motion, tides, and planetary orbits.

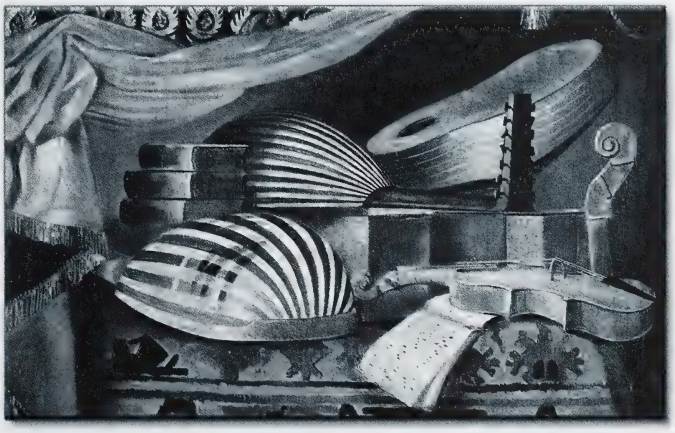
Turn to "CONNECTING MATH AND ASTRONOMY" on page 328 of *MATHPOWER™ 11* and read "Gravitational Forces."

Answer questions 1 to 11 of Investigation 1, "The Earth and the Moon" on pages 328 and 329 of the textbook.

Compare your responses with those in the Appendix, Section 3: Follow-up Activities, Enrichment.

## Conclusion

In this section, you investigated three types of functions: absolute value functions, rational functions, and radical functions. You used your graphing calculator to help sketch and describe these curves, to obtain graphical solutions to equations and inequalities involving these functions, and to support and visualize your algebraic solutions. In each activity, you applied these functions to solve real-world problems.



Applications of absolute value functions included the analysis of launch windows for satellites, temperatures at which certain chemical compounds are liquid, and in the variations in the distance of Earth from the Sun. You applied rational functions to solve problems involving driving speeds, proportions for mixtures and solutions, and measurements. Also, you used radical functions to model the pitch of guitar or violin strings, the speed of tsunami waves, and the period of a pendulum.

These are just a few examples of how absolute value, rational, and radical functions help determine a wide variety of information in the world today. You will encounter others in future mathematics courses.

## Assignment

Now, turn to Assignment Booklet 4B to complete the assignment for Section 3.

# Module Summary

Your target in this module was the function—a central concept in mathematics. In Section 1, you concentrated on polynomial functions and their graphs. You then solved equations and inequalities arising from polynomial functions and their applications. In Section 2, you delved into the algebra of functions, enhancing your general skills and empowering you to handle a broader variety of functions. Finally, in Section 3, you explored three new categories of functions: absolute value functions, rational functions, and the radical functions. Like polynomial functions, these functions led to equations, inequalities, and real-world applications.

Have you ever watched skilled archers concentrate? Or have you seen artists study their subjects, drawing sketch after sketch, before finally putting brush to canvas? Attention to detail, focus, and a definite goal are necessary for the archer, the artist, and the serious student of mathematics. Have you met the target you set for yourself in the module?

## Final Module Assignment

Now, turn to Assignment Booklet 4B to complete the Final Module Assignment.

# Glossary
# Suggested Answers

# Glossary

**absolute value:** the distance a real number, $x$, lies from the origin on the number line, denoted by $|x|$

**absolute value equation:** an equation with a variable within an absolute value symbol

**absolute value function:** a function with a variable within an absolute value symbol

**absolute value inequality:** an inequality that has a variable within an absolute value symbol

**asymptote:** a line a graph approaches more and more closely; a tangent at infinity

**composition of $f$ with $g$:** the function obtained by replacing $x$ with $g(x)$ in $y = f(x)$, written $f \circ g$ or $f(g(x))$

**continuous graph:** a graph with no breaks in the curve

**cubic equation:** a polynomial equation of degree 3

**cubic function:** a polynomial function of degree 3

**discontinuous graph:** a graph with a break in the curve

**extraneous root:** a number obtained in the process of solving an equation that does not satisfy the equation

**Factor Theorem:** a theorem stating that $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$

Conversely, if $P(b) = 0$, then $x - b$ is a factor of $P(x)$.

**Integral Zero Theorem:** a theorem stating that if $x = b$ is an integral zero of a polynomial, $P(x)$, with integral coefficients, then $b$ is a factor of the constant term

**inverse of a function:** a relation formed by interchanging the coordinates within each ordered pair of the original function

**inverse of a relation:** a relation formed by interchanging the coordinates within each ordered pair of the original relation

**polynomial equation:** the equation formed when a polynomial is set equal to 0

**polynomial function:** a function of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0, \text{ where } n \in W$$

**polynomial inequality:** an inequality containing a polynomial expression

**quartic function:** a polynomial function of degree 4

**radical equation:** an equation with a variable within a radicand

**radical function:** a function with a variable within a radicand

**radical inequality:** an inequality with a variable within a radicand

**rational equation:** an equation formed by setting a rational expression equal to 0

**rational function:** a function of the form $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$

**rational inequality:** an inequality containing a rational expression

**Rational Zero Theorem:** a theorem stating that if $\frac{b}{a}$ is a rational zero of a polynomial with integral coefficients, then the numerator $b$ is a factor of the polynomial's constant term and $a$ is a factor of the coefficient of the highest-degree term

**Remainder Theorem:** a theorem stating that when the polynomial $P(x)$ is divided by $x-b$, the remainder equals $P(b)$

**synthetic division:** a simpler, quicker method of dividing a polynomial by a binomial of the form $x-b$

# Suggested Answers

## Section 1: Activity 1

**1. a.** $h(5)=\frac{1}{6}(5)^3+\frac{1}{2}(5)^2+\frac{1}{3}(5)$

$=\frac{125}{6}+\frac{25}{2}+\frac{5}{3}$

$=\frac{125}{6}+\frac{75}{6}+\frac{10}{6}$

$=\frac{210}{6}$

$=35$

There are 35 cannonballs.

**b.** $h(10)=\frac{1}{6}(10)^3+\frac{1}{2}(10)^2+\frac{1}{3}(10)$

$=\frac{1000}{6}+\frac{100}{2}+\frac{10}{3}$

$=\frac{1000}{6}+\frac{300}{6}+\frac{20}{6}$

$=\frac{1320}{6}$

$=220$

There are 220 cannonballs.

**2. a. Textbook questions 1 to 6 of "Explore: Discover the Relationship," p. 198**

| | Polynomial, $P(x)$ | Divisor, $x-b$ | Quotient | Remainder | $P(b)$ |
|---|---|---|---|---|---|
| 1. | $x^2-7x+16$ | $x-3$ | $x-4$ | 4 | $P(3)=4$ |
| 2. | $2x^2+3x-8$ | $x-2$ | $2x+7$ | 6 | $P(2)=6$ |
| 3. | $3x^2+8x+4$ | $x+2$ | $3x+2$ | 0 | $P(-2)=0$ |
| 4. | $x^3+3x^2-3x-2$ | $x-1$ | $x^2+4x+1$ | $-1$ | $P(1)=-1$ |
| 5. | $x^3-6x-6$ | $x+2$ | $x^2-2x-2$ | $-2$ | $P(-2)=-2$ |
| 6. | $2x^3-x^2-2x+3$ | $x+1$ | $2x^2-3x+1$ | 2 | $P(-1)=2$ |

## Section 1: Activity 1 (continued)

**b. Textbook questions 1 to 5 of "Inquire," p. 198**

**1.** When each polynomial is divided by $x-b$, the remainder is equal to the value of $P(b)$.

**2.** **a.** Remainder $= P(4)$

**b.** Remainder $= P(-3)$

**3.** **a.** When $x^3+2x^2-7x-2$ is divided by $x-1$, the

$$\begin{aligned}\text{Remainder} &= P(1)\\ &= 1^3+2(1)^2-7(1)-2\\ &= 1+2-7-2\\ &= -6\end{aligned}$$

**b.** When $x^3+2x^2-7x-2$ is divided by $x+1$, the

$$\begin{aligned}\text{Remainder} &= P(-1)\\ &= (-1)^3+2(-1)^2-7(-1)-2\\ &= -1+2+7-2\\ &= 6\end{aligned}$$

**c.** When $x^3+2x^2-7x-2$ is divided by $x-2$, the

$$\begin{aligned}\text{Remainder} &= P(2)\\ &= 2^3+2(2)^2-7(2)-2\\ &= 8+8-14-2\\ &= 0\end{aligned}$$

**d.** When $x^3+2x^2-7x-2$ is divided by $x+3$, the

$$\begin{aligned}\text{Remainder} &= P(-3)\\ &= (-3)^3+2(-3)^2-7(-3)-2\\ &= -27+18+21-2\\ &= 10\end{aligned}$$

**4.** When a divisor is a factor of a polynomial, the remainder is 0.

**5. a.** Remainder $= h(1)$

$$\begin{aligned} &= -5(1)^2 + 8(1) + 2 \\ &= -5 + 8 + 2 \\ &= 5 \end{aligned}$$

When the polynomial that represents the height of the shot, $-5t^2 + 8t + 2$, is divided by $t - 1$, the remainder is 5.

**b.** The remainder represents the height of the shot one second after it was released.

**3. a.**

| $x-(-2)$ | $1x^3+0x^2-1x+7$ | | | |
|---|---|---|---|---|
| $-2$ | 1 | 0 | $-1$ | 7 |
| | | $-2$ | 4 | $-6$ |
| | 1 | $-2$ | 3 | 1 ← remainder |
| | quotient | | | |

The quotient is $x^2 - 2x + 3$, and the remainder is 1.

**b.**

| $x-1$ | $4x^2+0x-7$ | | |
|---|---|---|---|
| 1 | 4 | 0 | $-7$ |
| | | 4 | 4 |
| | 4 | 4 | $-3$ ← remainder |
| | quotient | | |

The quotient is $4x + 4$, and the remainder is $-3$.

**4.** The number of cannonballs $= h(3) = R$, where $R$ is the remainder when the polynomial $h(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$ is divided by $x - 3$.

Use synthetic division.

| 3 | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | 0 ← coefficents of $h(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x + 0$ |
|---|---|---|---|---|
| | | $\frac{1}{2}$ | 3 | 10 |
| | $\frac{1}{6}$ | 1 | $\frac{10}{3}$ | 10 ← remainder |

$\therefore h(3) = 10$ ← Remainder Theorem

There are 10 cannonballs in a pile 3 layers high.

## Section 1: Activity 1 (continued)

**5. a. Textbook questions 8, 16, 22, 26, 30, 39, 47, 50, 51, 52, and 53 of "Practice," pp. 202 and 203**

**8.** $P(x)=x^3+x^2+x-3$

$$\begin{aligned}P(-3)&=(-3)^3+(-3)^2+(-3)-3\\&=-27+9-3-3\\&=-24\end{aligned}$$

**16.** $g(y)=2y^3-3y^2+5$

$$g\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3-3\left(\frac{1}{2}\right)^2+5$$

$$=\cancel{2}^{1}\left(\frac{1}{\cancel{8}_{4}}\right)-3\left(\frac{1}{4}\right)+5$$

$$=\frac{1}{4}-\frac{3}{4}+5$$

$$=4\frac{2}{4}$$

$$=4\frac{1}{2}$$

**22.** From the Remainder Theorem, when $2x^3+3x^2-9x-10$, $P(x)$, is divided by $x-2$,

$$\begin{aligned}R&=P(2)\\&=2(2)^3+3(2)^2-9(2)-10\\&=2(8)+3(4)-18-10\\&=16+12-18-10\\&=0\end{aligned}$$

**26.** From the Remainder Theorem, when $2x^3-x^2-4x-4$, $P(x)$, is divided by $x+3$,

$$\begin{aligned}R&=P(-3)\\&=2(-3)^3-(-3)^2-4(-3)-4\\&=2(-27)-9+12-4\\&=-54-9+12-4\\&=-55\end{aligned}$$

**30.** From the Remainder Theorem, when $x^3+6x^2-3x-x+8$, $P(x)$, is divided by $x+1$,

$$\begin{aligned} R &= P(-1) \\ &= (-1)^3+6(-1)^2-3(-1)-(-1)+8 \\ &= -1+6-3+1+8 \\ &= 11 \end{aligned}$$

**39.** From the Remainder Theorem, when $3x^2-\sqrt{2}x+3$, $P(x)$, is divided by $x+\sqrt{2}$,

$$\begin{aligned} R &= P\left(-\sqrt{2}\right) \quad \longleftarrow \text{The divisor is } x+\sqrt{2}=x-\left(-\sqrt{2}\right). \\ &= 3\left(-\sqrt{2}\right)^2-\sqrt{2}\left(-\sqrt{2}\right)+3 \\ &= 3(2)+2+3 \\ &= 11 \end{aligned}$$

**47.** Apply the Remainder Theorem.

$$\begin{aligned} R &= P(-2) \quad \longleftarrow \text{The divisor is } x+2=x-(-2). \\ 3 &= k(-2)^2+3(-2)+1 \\ 3 &= 4k-6+1 \\ 3 &= 4k-5 \\ 8 &= 4k \\ 4k &= 8 \\ k &= 2 \end{aligned}$$

**50.** Apply the Remainder Theorem.

$$\begin{aligned} R &= P(-1) \quad \longleftarrow \text{The divisor is } x+1=x-(-1). \\ 3 &= (-1)^3+k(-1)^2+(-1)+2 \\ 3 &= -1+k-1+2 \\ 3 &= k \\ k &= 3 \end{aligned}$$

## Section 1: Activity 1 (continued)

**51.** When $4x^3 + mx^2 + nx + 11$ is divided by $x + 2$, the remainder is $-7$.

$$\therefore R = P(-2)$$
$$-7 = 4(-2)^3 + m(-2)^2 + n(-2) + 11$$
$$-7 = 4(-8) + 4m - 2n + 11$$
$$-7 = -32 + 4m - 2n + 11$$
$$-7 = -21 + 4m - 2n$$
$$14 = 4m - 2n$$
$$4m - 2n = 14$$
$$2m - n = 7 \quad (1)$$

When $4x^3 + mx^2 + nx + 11$ is divided by $x - 1$, the remainder is 14.

$$\therefore R = P(1)$$
$$14 = 4(1)^3 + m(1)^2 + n(1) + 11$$
$$14 = 4 + m + n + 11$$
$$14 = 15 + m + n$$
$$-1 = m + n$$
$$m + n = -1 \quad (2)$$

Solve the system for $m$ and $n$.

$$\begin{array}{rcll} 2m - n &=& 7 & (1) \\ m + n &=& -1 & (2) \\ \hline 3m &=& 6 & (1) + (2) \end{array}$$
$$m = 2$$

Substitute $m = 2$ into (2).

$$m + n = -1$$
$$2 + n = -1$$
$$n = -3$$

Thus, $m = 2$ and $n = -3$.

**52.** When $px^3 - x^2 + qx - 2$ is divided by $x - 1$, the remainder is 0.

$$\therefore R = P(1)$$
$$0 = p(1)^3 - (1)^2 + q(1) - 2$$
$$0 = p - 1 + q - 2$$
$$3 = p + q$$
$$p + q = 3 \quad (1)$$

When $px^3 - x^2 + qx - 2$ is divided by $x + 2$, the remainder is $-18$.

$$\therefore R = P(-2)$$
$$-18 = p(-2)^3 - (-2)^2 + q(-2) - 2$$
$$-18 = -8p - 4 - 2q - 2$$
$$-18 = -6 - 8p - 2q$$
$$-12 = -8p - 2q$$
$$-6 = -4p - q$$
$$-4p - q = -6 \quad (2)$$

Solve the system for $p$ and $q$.

$$\begin{array}{rrll} p+q & = & 3 & (1) \\ -4p-q & = & -6 & (2) \\ \hline -3p & = & -3 & (1)+(2) \end{array}$$
$$p = 1$$

Substitute $p = 1$ into (1).

$$p + q = 3$$
$$1 + q = 3$$
$$q = 2$$

Thus, $p = 1$ and $q = 2$.

## Section 1: Activity 1 (continued)

**53.** When $3x^3 + vx^2 - 5x + w$ is divided by $x+2$, the remainder is $-1$.

$$\therefore R = P(-2)$$
$$-1 = 3(-2)^3 + v(-2)^2 - 5(-2) + w$$
$$-1 = 3(-8) + 4v + 10 + w$$
$$-1 = -24 + 4v + 10 + w$$
$$-1 = -14 + 4v + w$$
$$13 = 4v + w$$
$$4v + w = 13 \quad (1)$$

When $3x^3 + vx^2 - 5x + w$ is divided by $x-3$, the remainder is 109.

$$\therefore R = P(3)$$
$$109 = 3(3)^3 + v(3)^2 - 5(3) + w$$
$$109 = 3(27) + 9v - 15 + w$$
$$109 = 81 + 9v - 15 + w$$
$$109 = 66 + 9v + w$$
$$43 = 9v + w$$
$$9v + w = 43 \quad (2)$$

Solve the system for $v$ and $w$.

$$\begin{array}{rrll} 4v + w & = & 13 & (1) \\ 9v + w & = & 43 & (2) \\ \hline -5v & = & -30 & (1) - (2) \end{array}$$
$$v = 6$$

Substitute $v = 6$ into (1).

$$4v + w = 13$$
$$4(6) + w = 13$$
$$24 + w = 13$$
$$w = -11$$

Thus, $v = 6$ and $w = -11$.

**b. Textbook questions 55 and 58 of "Applications and Problem Solving," p. 203**

**55.** When $ka^3 - 3a^2 + 5a - 8$ is divided by $a - 2$, the remainder is 22.

$$\therefore R = P(2)$$
$$22 = k(2)^3 - 3(2)^2 + 5(2) - 8$$
$$22 = 8k - 12 + 10 - 8$$
$$22 = 8k - 10$$
$$32 = 8k$$
$$8k = 32$$
$$k = 4$$

Substitute $k = 4$ into the original polynomial.

$$P(a) = ka^3 - 3a^2 + 5a - 8$$
$$= 4a^3 - 3a^2 + 5a - 8$$

Now, when $4a^3 - 3a^2 + 5a - 8$ is divided by $a + 1$,

$$R = P(-1)$$
$$= 4(-1)^3 - 3(-1)^2 + 5(-1) - 8$$
$$= -4 - 3 - 5 - 8$$
$$= -20$$

The remainder is $-20$.

## Section 1: Activity 1 (continued)

**58. a.** From the Remainder Theorem, when $h(d) = 0.0003d^2 + 2$ is divided by $d - 500$,

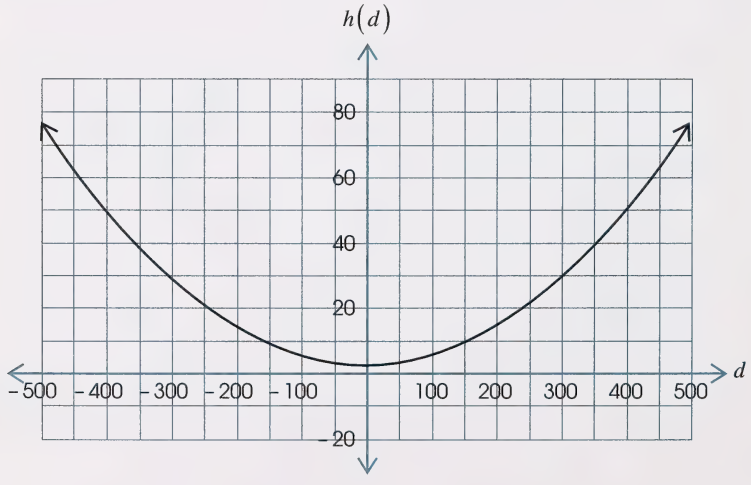
$$\begin{aligned} R &= h(500) \\ &= 0.0003(500)^2 + 2 \\ &= 75 + 2 \\ &= 77 \end{aligned}$$

**b.** From the Remainder Theorem, when $h(d) = 0.0003d^2 + 2$ is divided by $d + 500$,

$$\begin{aligned} R &= h(-500) \\ &= 0.0003(-500)^2 + 2 \\ &= 75 + 2 \\ &= 77 \end{aligned}$$

**c.** The results from questions 58.a. and 58.b. are equal.

The graph of $h(d) = 0.0003d^2 + 2$ is symmetric with respect to the $h(d)$-axis. Therefore, points lying the same distance on either side of the $h(d)$-axis will be located the same distance above the horizontal axis. In particular, $h(500) = h(-500)$.



**6. a. Textbook questions 41 and 42 of "Practice," p. 202**

**41.** From the Remainder Theorem, when $P(x) = 2x^2 + 5x + 7$ is divided by $2x - 3$,

$$\begin{aligned} R &= P\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) + 7 \\ &= \cancel{2}^{1}\left(\frac{9}{\cancel{4}_{2}}\right) + \frac{15}{2} + 7 \\ &= \frac{9}{2} + \frac{15}{2} + 7 \\ &= 19 \end{aligned}$$

**42.** From the Remainder Theorem, when $P(p) = 6p^2 + 5p - 4$ is divided by $3p + 4$,

$$\begin{aligned} R &= P\left(-\frac{4}{3}\right) \\ &= 6\left(-\frac{4}{3}\right)^2 + 5\left(-\frac{4}{3}\right) - 4 \\ &= \cancel{6}^{2}\left(\frac{16}{\cancel{9}_{3}}\right) - \frac{20}{3} - 4 \\ &= \frac{32}{3} - \frac{20}{3} - 4 \\ &= 0 \end{aligned}$$

## Section 1: Activity 1 (continued)

**b. Textbook questions 54, 62, and 63.a. of "Applications and Problem Solving," p. 203**

**54.** When $6y^3 - 3y^2 + 2y + 7$ is divided by $2y - 1$,

$$\begin{aligned} R &= P\left(\frac{1}{2}\right) \\ &= 6\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 7 \\ &= \overset{3}{\cancel{6}}\left(\frac{1}{\underset{4}{\cancel{8}}}\right) - 3\left(\frac{1}{4}\right) + 1 + 7 \\ &= \frac{3}{4} - \frac{3}{4} + 1 + 7 \\ &= 8 \end{aligned}$$

Therefore, when $2x^3 + 4x^2 - kx + 5$ is divided by $x + 3$, the remainder is 8.

$$\begin{aligned} \therefore\ R &= P(-3) \\ 8 &= 2(-3)^3 + 4(-3)^2 - k(-3) + 5 \\ 8 &= 2(-27) + 4(9) + 3k + 5 \\ 8 &= -54 + 36 + 3k + 5 \\ 8 &= 3k - 13 \\ 21 &= 3k \\ 3k &= 21 \\ k &= 7 \end{aligned}$$

**62. a.** Answers may vary. A sample prediction is given.

**Prediction:** "When $P(x)$ is divided by $b - x$, the quotient and remainder will be $-Q(x)$ and $-R$ respectively."

**b.** Suppose $P(x)=x^2-6x-3$.

First, divide $P(x)$ by $x-2$ using synthetic division.

$$\begin{array}{c|ccc} 2 & 1 & -6 & -3 \\ & & 2 & -8 \\ \hline & 1 & -4 & -11 \end{array}$$

In this case, $Q(x)=x-4$ and $R=-11$.

Next, divide $P(x)$ by $2-x$ or $(-x+2)$.

$$\begin{array}{r} -x+4 \\ -x+2\overline{)\,x^2-6x-3} \\ \underline{x^2-2x} \\ -4x-3 \\ \underline{-4x+8} \\ -11 \end{array}$$

In this case, the quotient, $-x+4$, equals $-Q(x)$ and the remainder, $-11$, equals $R$.

$$\begin{aligned} -x+4 &= -(x-4) \\ &= -Q(x) \end{aligned}$$

The prediction for the quotient was correct. The new quotient, obtained by dividing $P(x)$ by $2-x$, is opposite in sign to the original quotient, obtained by dividing $P(x)$ by $x-2$. However, the prediction for the new remainder was incorrect. The remainder did not change.

**c.** If $P(x)$ is divided by $x-b$ to give a quotient $Q(x)$ and a remainder $R$, then $P(x)=(x-b)\times Q(x)+R$. You can transform the division algorithm into an equivalent statement involving the divisor $b-x$.

$$\begin{aligned} P(x) &= (x-b)\times Q(x)+R \\ &= (-1)(x-b)\times(-1)Q(x)+R \\ &= \big[(-1)(x-b)\big]\times\big[(-1)Q(x)\big]+R \\ &= (-x+b)\times\big(-Q(x)\big)+R \\ &= (b-x)\times\big(-Q(x)\big)+R \end{aligned}$$

$R$ ← remainder

divisor: $b-x$ ; quotient: $-Q(x)$

Note: $(-1)(-1)=1$

Therefore, if $P(x)$ is divided by $b-x$, the quotient is $-Q(x)$ and the remainder is $R$.

## Section 1: Activity 1 (continued)

**63. a.** Answers may vary. A sample answer is given.

If $P(x)=(x-b)\times Q(x)+R$, then

$$P(x)=(x-3)\times Q(x)-4$$

divisor ↑ ↑ remainder

Choose any linear expression, $Q(x)$.

If $Q(x)=2x+5$, then

*Note: Quadratic means "of degree 2."*

$$\begin{aligned}P(x)&=(x-3)(2x+5)-4\\&=2x^2+5x-6x-15-4\\&=2x^2-x-19\end{aligned}$$

**Check**

From the Remainder Theorem, if $P(x)$ is correct, then $-4=P(3)$.

| LS | RS |
|---|---|
| $-4$ | $P(3)$ |
| | $=2(3)^2-3-19$ |
| | $=18-3-19$ |
| | $=-4$ |

LS = RS

Thus, a quadratic polynomial that gives a remainder of $-4$ when it is divided by $x-3$ is $2x^2-x-19$.

**7. Textbook question 6 of "Using the Strategies," p. 235**

**6.** Find the values of $A$, $B$, and $C$ by solving a system of equations.

**Row 1:** $3A+B+C=25$ (1)
**Row 2:** $A+3B+C=17$ (2)
**Row 3:** $A+2B+2C=20$ (3)

Find two equations where $C$ has been eliminated.

$$\begin{array}{rll} 3A+B+C=25 & (1) & \\ A+B-C= & (2) & \\ \hline 2A-2B=8 & (1)-(2) & \\ A-B=4 & (4) & \end{array} \qquad \begin{array}{rll} A+2B+2C=20 & (3) & \\ 2A+6B+2C=34 & (5):2\times(2) & \\ \hline -A-4B=-14 & (6):(3)-(5) & \end{array}$$

Solve for $A$.

$$\begin{array}{rll} A-B=4 & (4) & \\ -A-4B=-14 & (6) & \\ \hline -5B=-10 & (4)+(6) & \\ B=2 & & \end{array}$$

Substitute $B=2$ into (4).

$$\begin{aligned} A-B&=4\\ A-2&=4\\ A&=6 \end{aligned}$$

Substitute $A=6$ and $B=2$ into (1).

$$\begin{aligned} 3A+B+C&=25\\ 3(6)+2+C&=25\\ 18+2+C&=25\\ 20+C&=25\\ C&=5 \end{aligned}$$

To find $D$, substitute $A=6$, $B=2$, and $C=5$ into Column 1.

$$\begin{aligned} A+B+2C+D&=21\\ 6+2+2(5)+D&=21\\ 18+D&=21\\ D&=3 \end{aligned}$$

## Section 1: Activity 1 (continued)

The completed diagram should look as follows:

| | | | | | |
|---|---|---|---|---|---|
| 6 | 6 | 6 | 5 | 2 | 25 |
| 2 | 6 | 2 | 2 | 5 | 17 |
| 5 | 2 | 2 | 5 | 6 | 20 |
| 5 | 5 | 3 | 5 | 6 | 24 |
| 3 | 6 | 6 | 3 | 3 | **21** |
| 21 | 25 | 19 | 20 | **22** | |

## Section 1: Activity 2

**1. a. Textbook questions 1 to 12 of "Explore: Discover the Relationship," p. 204**

| | Polynomial, $P(x)$ | Divisor, $x-b$ | Remainder, $P(b)$ | Factor? (Yes or No) |
|---|---|---|---|---|
| **1.** | $x^3+x^2-4x-4$ | $x+1$ | 0 | Yes |
| **2.** | $x^3+x^2-4x-4$ | $x-1$ | $-6$ | No |
| **3.** | $x^3+x^2-4x-4$ | $x-2$ | 0 | Yes |
| **4.** | $x^3+x^2-4x-4$ | $x+2$ | 0 | Yes |
| **5.** | $x^3+x^2-4x-4$ | $x+3$ | $-10$ | No |
| **6.** | $x^3+2x^2-5x-6$ | $x+1$ | 0 | Yes |
| **7.** | $x^3+2x^2-5x-6$ | $x-2$ | 0 | Yes |
| **8.** | $x^3+2x^2-5x-6$ | $x-3$ | 24 | No |
| **9.** | $x^3+2x^2-5x-6$ | $x+3$ | 0 | Yes |
| **10.** | $x^3+2x^2-5x-6$ | $x+4$ | $-18$ | No |
| **11.** | $x^3+2x^2-5x-6$ | $x-5$ | 144 | No |
| **12.** | $x^3+2x^2-5x-6$ | $x+6$ | $-120$ | No |

**b. Textbook questions 1 to 4 of "Inquire," p. 205**

**1.** When a binomial is a factor of a polynomial, the constant term of the binomial must be a factor of the constant term of the polynomial. The constant term of the polynomial is obtained by multiplying the constant terms of the binomial factors of the polynomial.

**2.** If the constant term of a binomial is a factor of the constant term of the polynomial, the binomial is not necessarily a factor. For example, the constant term of $x+1$ is 1. Now, 1 is a factor of every integer, yet $x+1$ is not a factor of every integral polynomial.

The factors of $x^2-4$ are $x+2$ and $x-2$. The binomial $x+1$ does not divide $x^2-4$ evenly.

$$P(-1)=(-1)^2-4$$
$$=-3 \quad \longleftarrow \text{remainder}$$

**3. a.** The possible values of $b$ for the binomial factors, $x-b$, of $x^3+2x^2-13x+10$ are the factors of 10. They are $\pm1, \pm2, \pm5$, and $\pm10$.

**b.** Begin by testing 1 using synthetic division.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | −13 | 10 |
| | | 1 | 3 | −10 |
| | 1 | 3 | −10 | 0 ← remainder |
| | quotient | | | |

*Note: You are dividing by $x-1$ here.*

Because the remainder is 0, $x-1$ is a factor.

$$\therefore x^3+2x^2-13x+10=(x-1)(x^2+3x-10)$$
$$=(x-1)(x-2)(x+5)$$

**c.** $(x-1)(x-2)(x+5)=(x-1)(x^2+5x-2x-10)$

$$=(x-1)(x^2+3x-10)$$
$$=x^3+3x^2-10x-x^2-3x+10$$
$$=x^3+2x^2-13x+10$$

## Section 1: Activity 2 (continued)

**4. a.** $x^3 - 9x^2 + 20x = x\left(x^2 - 9x + 20\right)$

$= x(x-4)(x-5)$

The dimensions are $x$, $x-4$, and $x-5$.

**b.** If $x$ is a factor, then $P(0)=0$.

| LS | RS |
|---|---|
| $P(0)$ | 0 |
| $=0^3 - 9(0)^2 + 20(0)$ | |
| $=0-0+0$ | |
| $=0$ | |

LS = RS

*Note:* $x = x - 0$

If $x-4$ is a factor, then $P(4)=0$.

| LS | RS |
|---|---|
| $P(4)$ | 0 |
| $=4^3 - 9(4)^2 + 20(4)$ | |
| $=64-9(36)+80$ | |
| $=144-144$ | |
| $=0$ | |

LS = RS

If $x-5$ is a factor, then $P(5)=0$.

| LS | RS |
|---|---|
| $P(5)$ | 0 |
| $=5^3 - 9(5)^2 + 20(5)$ | |
| $=125-9(25)+100$ | |
| $=225-225$ | |
| $=0$ | |

LS = RS

**c.** $x=6$ $\quad x-4=6-4 \quad x-5=6-5$

$=2 \quad =1$

The dimensions of each rectangular prism are $1\text{ m}\times 2\text{ m}\times 6\text{ m}$.

**2. a. Textbook questions 1, 3, 7, 8, 22, 24, 26, 55, 58, and 61 of "Practice," pp. 209 and 210**

**1.** From the Factor Theorem, if $x-1$ is a factor of $P(x)$, then $P(1)=0$.

$\therefore P(x)=x^3-3x^2+4x-2$

$P(1)=1^3-3(1)^2+4(1)-2$

$=1-3+4-2$

$=0$

Therefore, $x-1$ is a factor of $P(x)=x^3-3x^2+4x-2$.

**3.** From the Factor Theorem, if $x-1$ is a factor of $P(x)$, then $P(1)=0$.

$\therefore P(x)=3x^3-x-3$

$P(1)=3(1)^3-1-3$

$=3-1-3$

$=-1$

Therefore, $x-1$ is not a factor of $P(x)=3x^3-x-3$.

**7.** From the Factor Theorem, if $x+2$ is a factor of $P(x)$, then $P(-2)=0$.

$\therefore P(x)=5x^2+2x+4$

$P(-2)=5(-2)^2+2(-2)+4$

$=5(4)-4+4$

$=20$

Therefore, $x+2$ is not a factor of $P(x)=5x^2+2x+4$.

## Section 1: Activity 2 (continued)

**8.** From the Factor Theorem, if $x+2$ is a factor of $P(x)$, then $P(-2)=0$.

$$\begin{aligned}\therefore P(x) &= x^3+2x^2-3x-6\\ P(-2) &= (-2)^3+2(-2)^2-3(-2)-6\\ &= -8+2(4)+6-6\\ &= 0\end{aligned}$$

Therefore, $x+2$ is a factor of $P(x)=x^3+2x^2-3x-6$.

**22.** From the Factor Theorem, if $x+3$ is a factor of $P(x)$, then $P(-3)=0$.

$$\begin{aligned}\therefore P(x) &= x^3+7x^2+17x+15\\ P(-3) &= (-3)^3+7(-3)^2+17(-3)+15\\ &= -27+7(9)-51+15\\ &= 0\end{aligned}$$

Therefore, $x+3$ is a factor of $P(x)=x^3+7x^2+17x+15$.

**24.** From the Factor Theorem, if $x+5$ is a factor of $P(x)$, then $P(-5)=0$.

$$\begin{aligned}\therefore P(x) &= x^4+5x^3+2x^2+7x-15\\ P(-5) &= (-5)^4+5(-5)^3+2(-5)^2+7(-5)-15\\ &= 625+5(-125)+2(25)-35-15\\ &= 625-625+50-35-15\\ &= 0\end{aligned}$$

Therefore, $x+5$ is a factor of $P(x)=x^4+5x^3+2x^2+7x-15$.

**26.** From the Factor Theorem, if $x+1$ is a factor of $P(x)$, then $P(-1)=0$.

$$\begin{aligned}\therefore P(x) &= 6x^2-17\\ P(-1) &= 6(-1)^2-17\\ &= -11\end{aligned}$$

Therefore, $x+1$ is not a factor of $P(x)=6x^2-17$.

**55.** Find a factor by evaluating $P(x)$ for values of $x$ that equal the possible values of $b$ in $x-b$. The possible values of $b$ are the factors of $-6$, which are $\pm 1, \pm 2, \pm 3$, and $\pm 6$.

Try $P(1)$.

$$\begin{aligned} P(x) &= x^3 - 6x^2 + 11x - 6 \\ P(1) &= 1^3 - 6(1)^2 + 11(1) - 6 \\ &= 1 - 6 + 11 - 6 \\ &= 0 \end{aligned}$$

Therefore, $x-1$ is a factor of $P(x) = x^3 - 6x^2 + 11x - 6$.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

Another factor is $x^2 - 5x + 6$.

$$\begin{aligned} \therefore P(x) &= x^3 - 6x^2 + 11x - 6 \\ &= (x-1)\left(x^2 - 5x + 6\right) \\ &= (x-1)(x-2)(x-3) \end{aligned}$$

**58.** Find a factor by evaluating $P(x)$ for values of $x$ that equal the possible values of $b$ in $x-b$. The possible values of $b$ are the factors of $-3$, which are $\pm 1$ and $\pm 3$.

Try $P(-3)$.

$$\begin{aligned} P(x) &= x^3 + 4x^2 + 2x - 3 \\ P(-3) &= (-3)^3 + 4(-3)^2 + 2(-3) - 3 \\ &= -27 + 4(9) - 6 - 3 \\ &= -27 + 36 - 6 - 3 \\ &= 0 \end{aligned}$$

Therefore, $x+3$ is a factor of $P(x) = x^3 + 4x^2 + 2x - 3$.

## Section 1: Activity 2 (continued)

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} -3 & 1 & 4 & 2 & -3 \\ & & -3 & -3 & 3 \\ \hline & 1 & 1 & -1 & 0 \end{array}$$

Another factor is $x^2+x-1$.

$\therefore P(x)=(x+3)\left(x^2+x-1\right)$

The trinomial $x^2+x-1$ cannot be factored over the integers.

**61.** Find a factor by evaluating $P(x)$ for values of $x$ that equal the possible values of $b$ in $x-b$. The possible values of $b$ are the factors of $-8$, which are $\pm1, \pm2, \pm4$, and $\pm8$.

Try $P(4)$.

$$\begin{aligned} P(x)&=x^3-2x^2-6x-8 \\ P(4)&=4^3-2(4)^2-6(4)-8 \\ &=64-2(16)-24-8 \\ &=64-32-24-8 \\ &=0 \end{aligned}$$

Therefore, $x-4$ is a factor of $P(x)=x^3-2x^2-6x-8$.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} 4 & 1 & -2 & -6 & -8 \\ & & 4 & 8 & 8 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

Another factor is $x^2+2x+2$.

$\therefore P(x)=(x-4)\left(x^2+2x+2\right)$

The trinomial $x^2+2x+2$ cannot be factored over the integers.

**b. Textbook questions 75, 78.a., 82.a., 85, and 86 of "Applications and Problem Solving," pp. 210 and 211**

**75. a.** $V(h) = h^3 - 2h^2 + h$

$= h(h^2 - 2h + 1)$

$= h(h-1)(h-1)$

**b.** If $h = 1.5$, then $h - 1 = 1.5 - 1$

$= 0.5$

The dimensions of the cabinet are 1.5 m × 0.5 m × 0.5 m.

**78. a.** Find a factor by evaluating $P(x)$ for values of $x$ that equal the possible values of $b$ in $x - b$. The possible values of $b$ are the factors of $-24$, which are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12$, and $\pm 24$.

Try $P(-1)$.

$P(x) = x^4 + 4x^3 - 7x^2 - 34x - 24$

$P(-1) = (-1)^4 + 4(-1)^3 - 7(-1)^2 - 34(-1) - 24$

$= 1 - 4 - 7 + 34 - 24$

$= 0$

Therefore, $x + 1$ is a factor of $P(x) = x^4 + 4x^3 - 7x^2 - 34x - 24$.

Use synthetic division to find another factor.

| | | | | | |
|---|---|---|---|---|---|
| $-1$ | 1 | 4 | $-7$ | $-34$ | $-24$ |
| | | $-1$ | $-3$ | 10 | 24 |
| | 1 | 3 | $-10$ | $-24$ | 0 |

Another factor is $x^3 + 3x^2 - 10x - 24$.

$\therefore P(x) = (x+1)(x^3 + 3x^2 - 10x - 24)$

Use the Factor Theorem to find a factor of $Q(x) = x^3 + 3x^2 - 10x - 24$.

Try $Q(3)$.

$Q(x) = x^3 + 3x^2 - 10x - 24$

$Q(3) = 3^3 + 3(3)^2 - 10(3) - 24$

$= 27 + 27 - 30 - 24$

$= 0$

## Section 1: Activity 2 (continued)

Therefore, $x-3$ is a factor of $Q(x)=x^3+3x^2-10x-24$.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 3 | −10 | −24 |
| | | 3 | 18 | 24 |
| | 1 | 6 | 8 | 0 |

Another factor is $x^2+6x+8$.

$$\therefore P(x)=(x+1)\left(x^3+3x^2-10x-24\right)$$
$$=(x+1)(x-3)\left(x^2+6x+8\right)$$
$$=(x+1)(x-3)(x+2)(x+4)$$

**82. a.** From the Factor Theorem, if $x-3$ is a factor of $P(x)$, then $P(3)=0$.

$$\therefore P(x)=kx^3-10x^2+2x+3$$
$$P(3)=k(3)^3-10(3)^2+2(3)+3$$
$$0=27k-90+6+3$$
$$0=27k-81$$
$$27k=81$$
$$k=3$$

**85.** From the Factor Theorem, if $x+1$ is a factor of $P(x)$, then $P(-1)=0$.

$$\therefore P(x)=6x^3+mx^2+nx-5$$
$$P(-1)=6(-1)^3+m(-1)^2+n(-1)-5$$
$$0=-6+m-n-5$$
$$0=m-n-11$$
$$11=m-n$$
$$m-n=11 \quad (1)$$

From the Remainder Theorem, if the remainder is $-4$ when $P(x)$ is divided by $x-1$, then $P(1)=-4$.

$$\therefore P(x)=6x^3+mx^2+nx-5$$
$$P(1)=6(1)^3+m(1)^2+n(1)-5$$
$$-4=6+m+n-5$$
$$-4=m+n+1$$
$$-5=m+n$$
$$m+n=-5 \quad (2)$$

Solve the following system for $m$ and $n$:

$$\begin{array}{rll} m-n= & 11 & (1) \\ m+n= & -5 & (2) \\ \hline 2m= & 6 & (1)+(2) \\ m= & 3 & \end{array}$$

Substitute $m=3$ into (1).

$$m-n=11$$
$$3-n=11$$
$$-n=8$$
$$n=-8$$

Therefore, $m=3$ and $n=-8$.

**86.** From the Factor Theorem, if $x-a$ is a factor of the polynomial $P(x)$, then $P(a)=0$.

$$\therefore P(x)=x^3-ax^2+bx^2-abx+cx-ac$$
$$P(a)=a^3-a(a)^2+b(a)^2-ab(a)+c(a)-ac$$
$$=a^3-a^3+a^2b-a^2b+ac-ac$$
$$=0$$

Therefore, $x-a$ is factor of $P(x)=x^3-ax^2+bx^2-abx+cx-ac$.

## Section 1: Activity 2 (continued)

**3. Textbook questions 43, 48, and 49 of "Practice," p. 210**

**43.** From the Factor Theorem, if $2x+1$ is a factor of $P(x)$, then $P\left(-\frac{1}{2}\right)=0$.

$$\therefore P(x)=2x^3+x^2+2x+1$$

$$P\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)+1$$

$$=\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{1}{4}-1+1$$

$$=-\frac{1}{4}+\frac{1}{4}-1+1$$

$$=0$$

Therefore, $2x+1$ is a factor of $P(x)=2x^3+x^2+2x+1$.

**48.** From the Factor Theorem, if $3x-2$ is a factor of $P(x)$, then $P\left(\frac{2}{3}\right)=0$.

$$\therefore P(x)=3x^4-2x^3+12x-8$$

$$P\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^4-2\left(\frac{2}{3}\right)^3+\frac{4}{1}12\left(\frac{2}{3}\right)-8$$

$$=\frac{1}{3}\left(\frac{16}{27}\right)-2\left(\frac{8}{27}\right)+8-8$$

$$=\frac{16}{27}-\frac{16}{27}+8-8$$

$$=0$$

Therefore, $3x-2$ is a factor of $P(x)=3x^4-2x^3+12x-8$.

**49.** From the Factor Theorem, if $2x-1$ is a factor of $P(x)$, then $P\left(\frac{1}{2}\right)=0$.

$$\therefore P(x)=8x^2+2x-1$$

$$P\left(\frac{1}{2}\right)=8\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)-1$$

$$=8\left(\frac{1}{4}\right)+1-1$$

$$=2+1-1$$

$$=2$$

Therefore, $2x-1$ is not a factor of $P(x)=8x^2+2x-1$.

**4. a. Textbook questions 65 and 74 of "Practice," p. 210**

**65.** If $2x^3-9x^2+10x-3$ has a factor of the form $ax-b$, then $b$ is a factor of the constant term, which is $-3$, and $a$ is a factor of the coefficient of $x^3$, which is 2.

The factors of $-3$ are $\pm1$ and $\pm3$; the factors of 2 are $\pm1$ and $\pm2$. Therefore, the possible ratios of $\frac{b}{a}$ are $\pm1, \pm\frac{1}{2}, \pm3$, and $\pm\frac{3}{2}$.

Try $P(1)$.

$$P(x)=2x^3-9x^2+10x-3$$

$$P(1)=2(1)^3-9(1)^2+10(1)-3$$

$$=2-9+10-3$$

$$=0$$

Therefore, $x-1$ is a factor of $P(x)=2x^3-9x^2+10x-3$.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | −9 | 10 | −3 |
| | | 2 | −7 | 3 |
| | 2 | −7 | 3 | 0 |

## Section 1: Activity 2 (continued)

Another factor is $2x^2 - 7x + 3$.

$$\therefore P(x) = (x-1)\left(2x^2 - 7x + 3\right)$$
$$= (x-1)(2x-1)(x-3)$$

**74.** If $4x^3 + 3x^2 - 4x - 3$ has a factor of the form $ax - b$, then $b$ is a factor of the constant term, which is $-3$, and $a$ is a factor of the coefficient of $x^3$, which is 4.

The factors of $-3$ are $\pm 1$ and $\pm 3$, and the factors of 4 are $\pm 1, \pm 2$, and $\pm 4$. Therefore, the possible ratios of $\frac{b}{a}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 3, \pm\frac{3}{2}$, and $\pm\frac{3}{4}$.

Try $P(1)$.

$$P(x) = 4x^3 + 3x^2 - 4x - 3$$
$$P(1) = 4(1)^3 + 3(1)^2 - 4(1) - 3$$
$$= 4 + 3 - 4 - 3$$
$$= 0$$

Therefore, $x-1$ is a factor of $P(x) = 4x^3 + 3x^2 - 4x - 3$.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 1 | 4 | 3 | −4 | −3 |
| | | 4 | 7 | 3 |
| | 4 | 7 | 3 | 0 |

Another factor is $4x^2 + 7x + 3$.

$$\therefore P(x) = (x-1)\left(4x^2 + 7x + 3\right)$$
$$= (x-1)(x+1)(4x+3)$$

**b. Textbook questions 79.a., 84, 89, and 93 of "Applications and Problem Solving," pp. 210 and 211**

**79. a.** If $8x^3+4x^2-2x-1$ has a factor of the form $ax-b$, then $b$ is a factor of the constant term, which is $-1$, and $a$ is a factor of the coefficient of $x^3$, which is 8.

The factors of $-1$ are $\pm1$; the factors of 8 are $\pm1, \pm2, \pm4$, and $\pm8$. Therefore, the possible ratios of $\frac{b}{a}$ are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$.

Try $P\left(-\frac{1}{2}\right)$.

$$P(x)=8x^3+4x^2-2x-1$$

$$P\left(-\frac{1}{2}\right)=8\left(-\frac{1}{2}\right)^3+4\left(-\frac{1}{2}\right)^2-2\left(-\frac{1}{2}\right)-1$$

$$=8\left(-\frac{1}{8}\right)+4\left(\frac{1}{4}\right)+1-1$$

$$=-1+1+1-1$$

$$=0$$

Therefore, $x+\frac{1}{2}$ is a factor of $P(x)=8x^3+4x^2-2x-1$.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} -\frac{1}{2} & 8 & 4 & -2 & -1 \\ & & -4 & 0 & 1 \\ \hline & 8 & 0 & -2 & 0 \end{array}$$

Another factor is $8x^2-2$.

$$\therefore P(x)=\left(x+\frac{1}{2}\right)\left(8x^2-2\right)$$

$$=\left(x+\frac{1}{2}\right)(2)\left(4x^2-1\right)$$

$$=(2x+1)\left(4x^2-1\right)$$

$$=(2x+1)(2x+1)(2x-1)$$

## Section 1: Activity 2 (continued)

**84. a.** Because $P(2)=0, x-2$ is a factor.

Because $P(-4)=0, x+4$ is a factor.

Because $P\left(-\frac{1}{3}\right)=0, 3x+1$ is a factor.

**b.** There are no other factors because $P(x)$ is a cubic polynomial; cubic polynomials have only three linear factors.

Check that $3x^3+7x^2-22x-8=(x-2)(x+4)(3x+1)$.

| LS | RS |
|---|---|
| $3x^3+7x^2-22x-8$ | $(x-2)(x+4)(3x+1)$ |
| | $=(x-2)\left(3x^2+x+12x+4\right)$ |
| | $=(x-2)\left(3x^2+13x+4\right)$ |
| | $=3x^3+13x^2+4x-6x^2-26x-8$ |
| | $=3x^3+7x^2-22x-8$ |

LS = RS

**89. a.** From the Factor Theorem, if $x+1$ is a factor of $x^{100}-1$, then $P(-1)=0$.

$$\therefore P(x)=x^{100}-1$$
$$P(-1)=(-1)^{100}-1$$
$$=1-1$$
$$=0$$

Therefore, $x+1$ is a factor of $x^{100}-1$.

From the Factor Theorem, if $x-1$ is a factor, then $P(1)=0$.

$$\therefore P(x)=x^{100}-1$$
$$P(1)=1^{100}-1$$
$$=1-1$$
$$=0$$

Therefore, $x-1$ is also a factor of $x^{100}-1$.

**b.** From the Factor Theorem, if $x+1$ is a factor of $x^{99}+1$, then $P(-1)=0$.

$$\begin{aligned}\therefore P(x) &= x^{99}+1\\ P(-1) &= (-1)^{99}+1\\ &= -1+1\\ &= 0\end{aligned}$$

Therefore, $x+1$ is a factor of $x^{99}+1$.

From the Factor Theorem, if $x-1$ is a factor of $x^{99}+1$, then $P(1)=0$.

$$\begin{aligned}\therefore P(x) &= x^{99}+1\\ P(1) &= 1^{99}+1\\ &= 1+1\\ &= 2\end{aligned}$$

Therefore, $x-1$ is not a factor of $x^{99}+1$.

**93. a.** From the Factor Theorem, if $x-1$ is a factor of $ax^3+bx^2+cx+d$, then $P(1)=0$.

$$\begin{aligned}\therefore P(x) &= ax^3+bx^2+cx+d\\ P(1) &= a(1)^3+b(1)^2+c(1)+d\\ 0 &= a+b+c+d\end{aligned}$$

Thus, $x-1$ is a factor, and the sum of the coefficients is 0.

**b.** $3+5-6-2=0$

Therefore, $x-1$ is a factor of $3x^3+5x^2-6x-2$.

**c.** $2-9-1-8=-16$

Therefore, $x-1$ is not a factor of $2x^3-9x^2-x-8$.

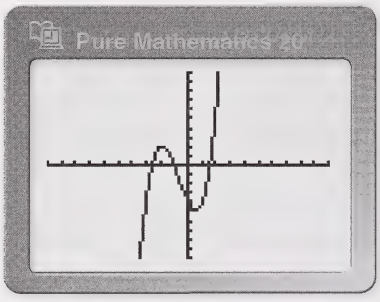
**d.** $-5+4+1=0$

Therefore, $x-1$ is a factor of $-5x^3+4x+1$.

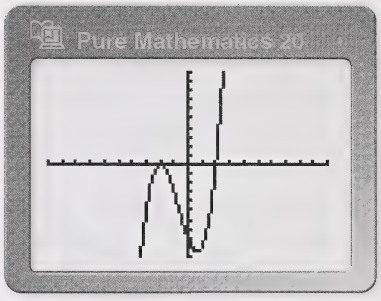
## Section 1: Activity 2 (continued)

**5. a. Textbook questions 1 to 9 of Investigation 1, "Cubic or Third-Degree Functions and Equations," p. 212**

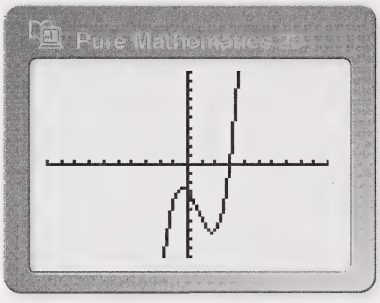
**1. a.** $y = x^3 + 2x^2 - 3x - 4$



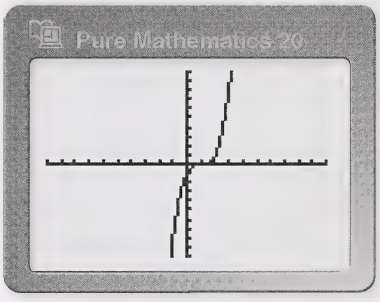

**b.** $y = x^3 + 2x^2 - 4x - 8$



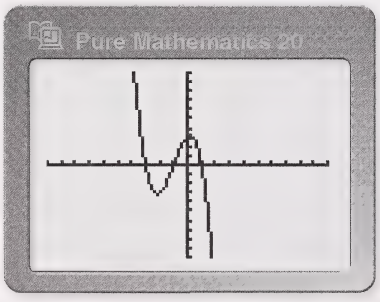

**c.** $y = x^3 - 2x^2 - 2x - 3$



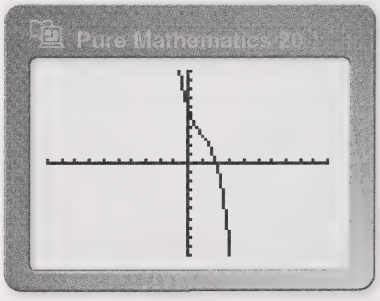

**d.** $y = x^3 - 3x^2 + 3x - 1$



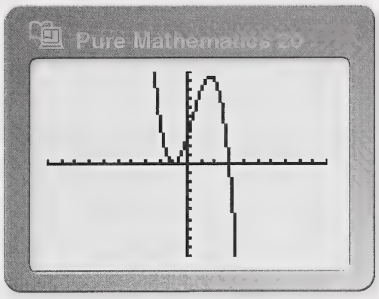

**e.** $y = -x^3 - 3x^2 + x + 3$



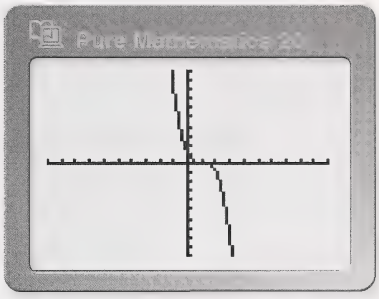

**f.** $y = -x^3 + 3x^2 - 5x + 6$



**g.** $y = -x^3 + x^2 + 5x + 3$



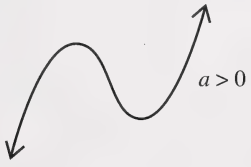

**h.** $y = -x^3 + 3x^2 - 3x + 1$




**2.** The shape of the graph of a cubic function is like the letter S.

**3.** A graph of a quadratic function is shaped like the letter U.

**4.** The graphs of cubic functions where $a$ is positive rise to the right.

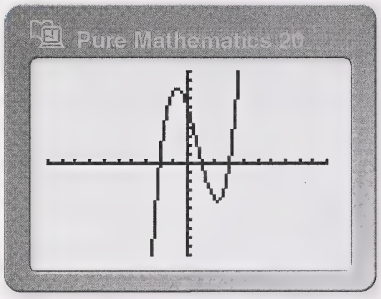


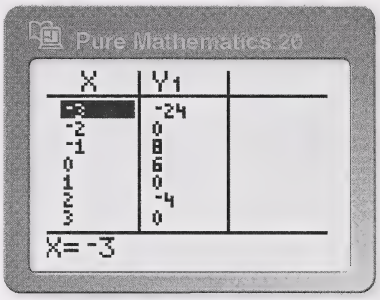

The graphs of cubic functions where $a$ is negative fall to the right.

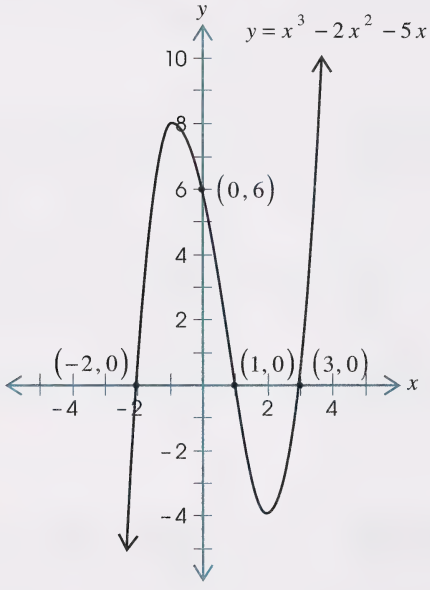
**5.** The value of $d$ in a cubic function represents the $y$-intercept.

## Section 1: Activity 2 (continued)

**6. a.** Graph $y = x^3 - 2x^2 - 5x + 6$ on your graphing calculator; then press 2nd [ TABLE ] to display the table of values.




The $x$-intercepts are $-2, 1$, and 3.

**b.** The $x$-intercepts are the zeros of the function $y = x^3 - 2x^2 - 5x + 6$; that is, they are the roots of $x^3 - 2x^2 - 5x + 6 = 0$.

*Note:*

$P(-2) = 0$

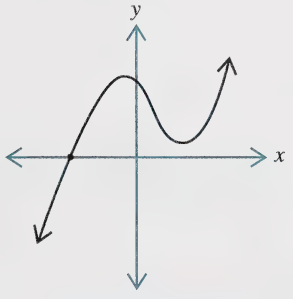
$P(1) = 0$

$P(3) = 0$

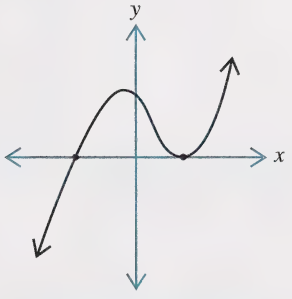
**7.**

| | Cubic Equation | Number of Distinct Real Roots |
|---|---|---|
| **a.** | $x^3+2x^2-3x-4=0$ | 3 |
| **b.** | $x^3+2x^2-4x-8=0$ | 2 |
| **c.** | $x^3-2x^2-2x-3=0$ | 1 |
| **d.** | $x^3-3x^2+3x-1=0$ | 1 |
| **e.** | $-x^3-3x^2+x+3=0$ | 3 |
| **f.** | $-x^3+3x^2-5x+6=0$ | 1 |
| **g.** | $-x^3+x^2+5x+3=0$ | 2 |
| **h.** | $-x^3+3x^2-3x+1=0$ | 1 |

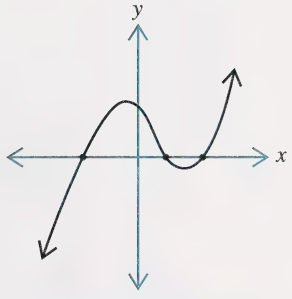
**8. a.** There could be 1, 2, or 3 distinct real roots.




1 Distinct Real Root

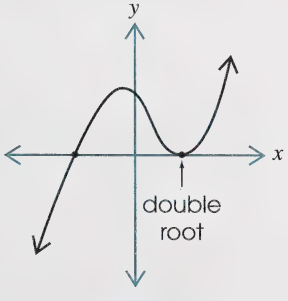



2 Distinct Real Roots

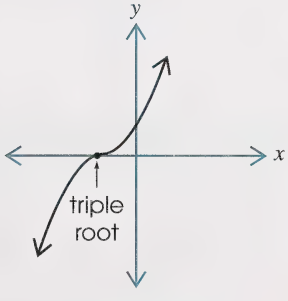



3 Distinct Real Roots

**b.** There could be 2 or 3 equal real roots.



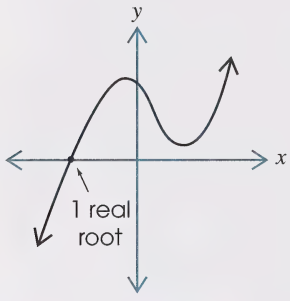

2 Equal Roots




3 Equal Roots

## Section 1: Activity 2 (continued)

**c.** There could be 2 imaginary roots.

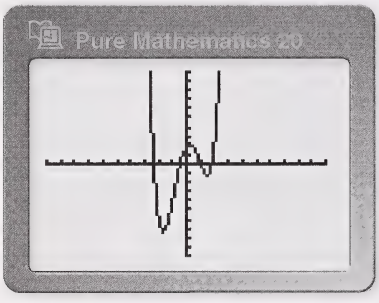



2 Imaginary Roots, 1 Real Root

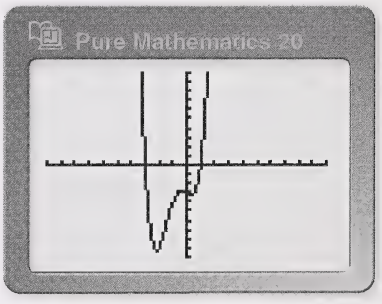
**9.** A cubic equation must have at least one real root because imaginary roots occur in pairs.

**b. Textbook questions 1 to 9 of Investigation 2, "Quartic or Fourth-Degree Functions and Equations," p. 213**

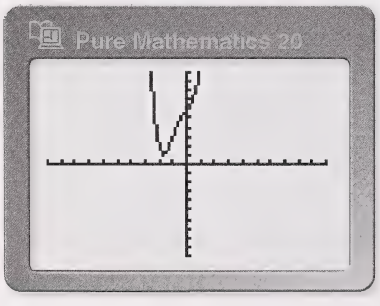
**1. a.** $y = x^4 - 5x^2 + 2x + 2$



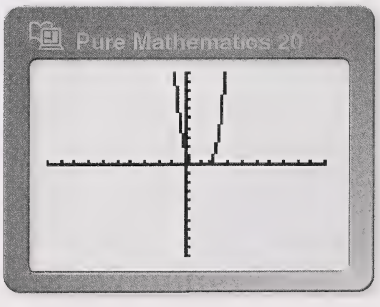

**b.** $y = x^4 + 3x^3 - x - 3$



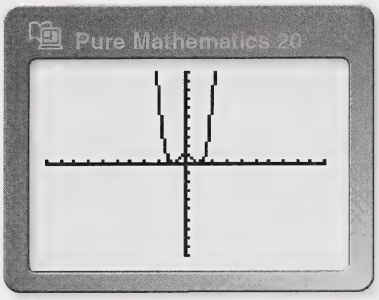

**c.** $y = x^4 + 2x^3 + 2x + 6$



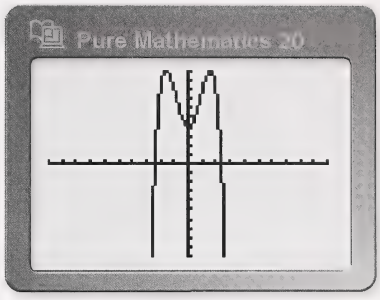

**d.** $y = x^4 - 4x^3 + 6x^2 - 4x + 1$



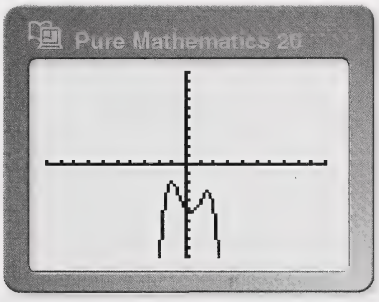
**e.** $y = x^4 - 2x^2 + 1$



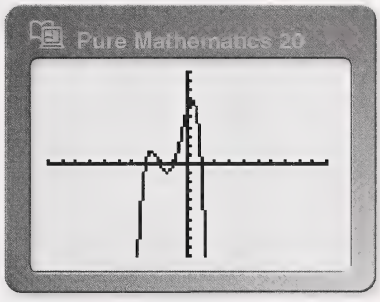

**f.** $y = -x^4 + 5x^2 + 4$



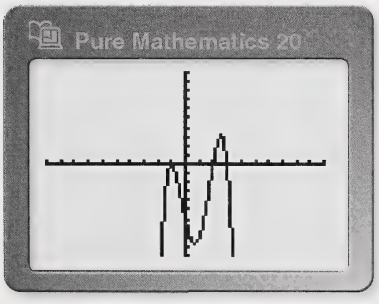

**g.** $y = -x^4 + x^3 + 3x^2 - 2x - 5$



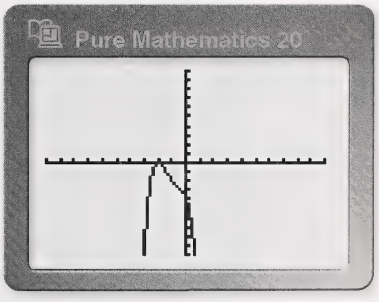

**h.** $y = -x^4 - 5x^3 - 5x^2 + 5x + 6$




**i.** $y = -x^4 + 3x^3 + 3x^2 - 7x - 6$




**j.** $y = -x^4 - 4x^3 - 5x^2 - 4x - 4$



## Section 1: Activity 2 (continued)

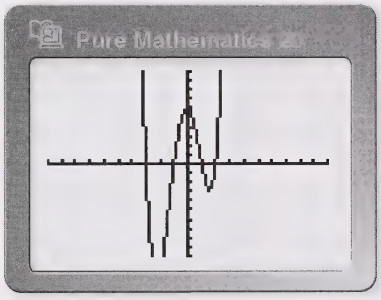
2. The graphs of quartic functions are shaped like the letter W, the letter M, or the letter U.

3. The graph of a cubic function is always shaped like the letter S; the graph of a quadratic function is always shaped like the letter U.

4. The graph of a quartic function where $a$ is positive opens upward. If $a$ is negative, the graph of a quartic function opens downward.

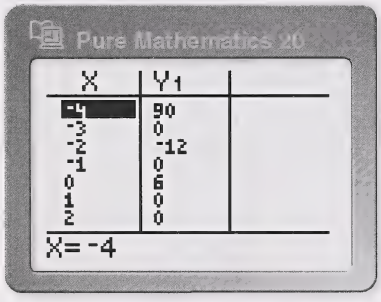
5. The value of $e$ represents the $y$-intercept.

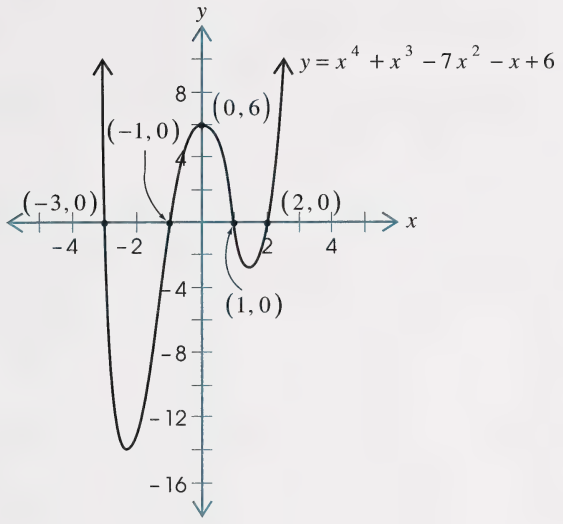
6. **a.** $y = x^4 + x^3 - 7x^2 - x + 6$




Because some of the graph is cut off when using the standard window settings, you will need to alter these settings.

Press 2nd [ TABLE ]






The $x$-intercepts are $-3, -1, 1$, and 2.

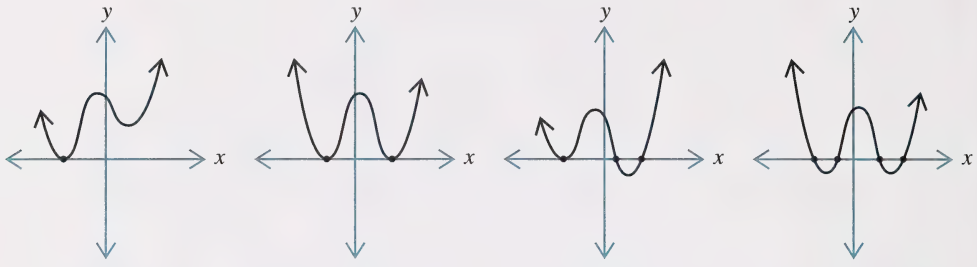
**b.** Because the $x$-intercepts of the graph are the zeros of the function, the $x$-intercepts are solutions of the equation $x^4 + x^3 - 7x^2 - x + 6 = 0$.

**7.**

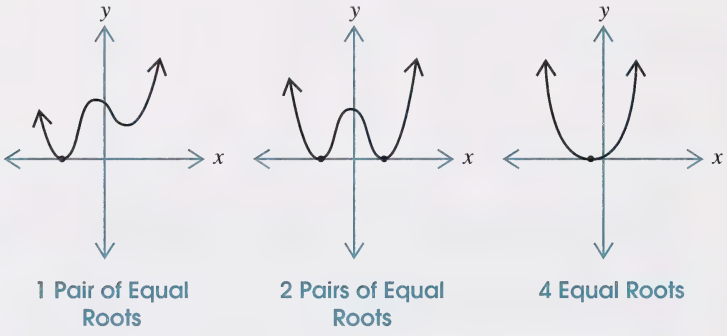
| | Quartic Equation | Number of Distinct Real Roots |
|---|---|---|
| **a.** | $x^4 - 5x^2 + 2x + 2 = 0$ | 4 |
| **b.** | $x^4 + 3x^3 - x - 3 = 0$ | 2 |
| **c.** | $x^4 + 2x^3 + 2x + 6 = 0$ | 0 |
| **d.** | $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$ | 1 |
| **e.** | $x^4 - 2x^2 + 1 = 0$ | 2 |
| **f.** | $-x^4 + 5x^2 + 4 = 0$ | 2 |
| **g.** | $-x^4 + x^3 + 3x^2 - 2x - 5 = 0$ | 0 |
| **h.** | $-x^4 - 5x^3 - 5x^2 + 5x + 6 = 0$ | 4 |
| **i.** | $-x^4 + 3x^3 + 3x^2 - 7x - 6 = 0$ | 3 |
| **j.** | $-x^4 - 4x^3 - 5x^2 - 4x - 4 = 0$ | 1 |

## Section 1: Activity 2 (continued)

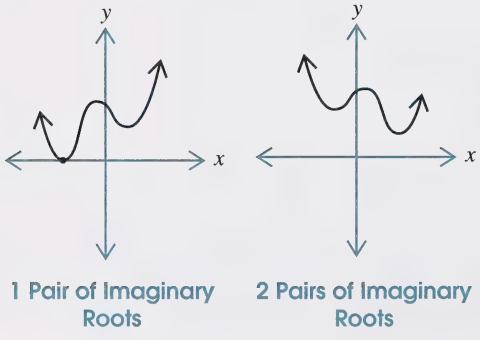
**8. a.** There could be 1, 2, 3, or 4 distinct real roots of a quartic equation.




**b.** There could be 1 or 2 pairs of equal roots, or there could be 4 equal roots.




**c.** There could be 1 or 2 pairs of imaginary roots.

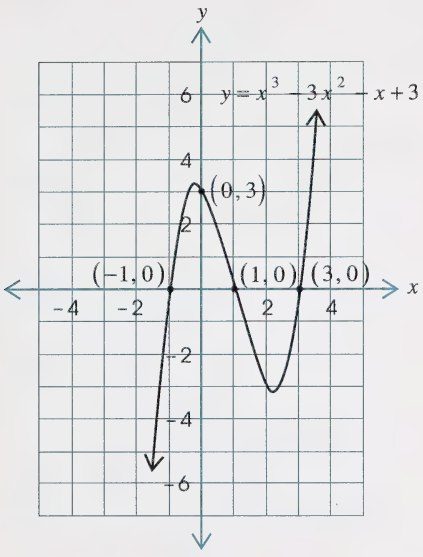



**9.** No, a quartic equation could have 2 different pairs of equal roots.

## Section 1: Activity 3

**1. a. Textbook questions a. to d. of "Explore: Interpret the Graph," p. 214**

**a.**




**b.** The $x$-intercepts are $-1, 1$, and 3.

**c.** The $y$-intercept is 3.

**d.** Refer to the answer to question a.

**b. Textbook questions 1 and 2 of "Inquire," p. 214**

**1.** The $x$-intercepts of the graph of $y = x^3 - 3x^2 - x + 3$ have $x$-coordinates that are solutions to the equation $x^3 - 3x^2 - x + 3 = 0$. To find $x$-intercepts, you must replace $y$ with 0 and solve the resulting equation.

**2. a.**
$$\begin{aligned} x^3 - 3x^2 - x + 3 &= \left(x^3 - 3x^2\right) - \left(x - 3\right) \\ &= x^2\left(x - 3\right) - 1\left(x - 3\right) \\ &= \left(x - 3\right)\left(x^2 - 1\right) \\ &= \left(x - 3\right)\left(x - 1\right)\left(x + 1\right) \end{aligned}$$

**b.** Equate each factor to 0 and solve the resulting equations to find the roots of the equation.

$$\begin{array}{lll} x - 3 = 0 & x - 1 = 0 & x + 1 = 0 \\ x = 3 & x = 1 & x = -1 \end{array}$$

The roots are $-1, 1$, and 3.

## Section 1: Activity 3 (continued)

**2. a. Textbook questions 4, 5, 9, 11, 13, 27, 35, and 42 of "Practice," pp. 219 and 220**

**4.** $(x+6)(x-3)^2=0$

$\therefore\ x+6=0 \quad \text{or} \quad x-3=0$

$x=-6 \qquad\qquad x=3$

The roots are $-6$ and 3.

**5.** $x^3+x^2-6x=0$

$x\left(x^2+x-6\right)=0$

$x(x+3)(x-2)=0$

$\therefore\ x=0 \quad \text{or} \quad x+3=0 \quad \text{or} \quad x-2=0$

$x=0 \qquad\qquad x=-3 \qquad\qquad x=2$

**Check**

For $x=0$,

| LS | RS |
|---|---|
| $x^3+x^2-6x$ | 0 |
| $=0^3+0^2-6(0)$ | |
| $=0$ | |

LS = RS

For $x=-3$,

| LS | RS |
|---|---|
| $x^3+x^2-6x$ | 0 |
| $=(-3)^3+(-3)^2-6(-3)$ | |
| $=-27+9+18$ | |
| $=0$ | |

LS = RS

For $x=2$,

| LS | RS |
|---|---|
| $x^3+x^2-6x$ | 0 |
| $=2^3+2^2-6(2)$ | |
| $=8+4-12$ | |
| $=0$ | |

LS = RS

Therefore, the roots are $-3, 0$, and 2.

**9.**
$$x^3+3x^2-x-3=0$$
$$\left(x^3+3x^2\right)+\left(-x-3\right)=0$$
$$x^2\left(x+3\right)-1\left(x+3\right)=0$$
$$\left(x+3\right)\left(x^2-1\right)=0$$
$$\left(x+3\right)\left(x-1\right)\left(x+1\right)=0$$

$\therefore\ x+3=0$ or $x-1=0$ or $x+1=0$

$x=-3$ $x=1$ $x=-1$

**Check**

For $x=-3$,

| | LS | RS |
|---|---|---|
| $x^3+3x^2-x-3$ | | 0 |
| $=(-3)^3+3(-3)^2-(-3)-3$ | | |
| $=-27+27+3-3$ | | |
| $=0$ | | |
| | LS | = RS |

For $x=1$,

| | LS | RS |
|---|---|---|
| $x^3+3x^2-x-3$ | | 0 |
| $=1^3+3(1)^2-1-3$ | | |
| $=1+3-1-3$ | | |
| $=0$ | | |
| | LS | = RS |

For $x=-1$,

| | LS | RS |
|---|---|---|
| $x^3+3x^2-x-3$ | | 0 |
| $=(-1)^3+3(-1)^2-(-1)-3$ | | |
| $=-1+3+1-3$ | | |
| $=0$ | | |
| | LS | = RS |

Therefore, the roots are $-3, -1$, and $1$.

**11.** Let $P(a)=a^3+2a^2-7a+4$, and use the Integral Zero Theorem to find a factor.

The factors of 4 are $\pm1, \pm2$, and $\pm4$.

## Section 1: Activity 3 (continued)

Try $P(1)$.

$$P(a) = a^3 + 2a^2 - 7a + 4$$
$$P(1) = 1^3 + 2(1)^2 - 7(1) + 4$$
$$= 1 + 2 - 7 + 4$$
$$= 0$$

Therefore, $a - 1$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | −7 | 4 |
| | | 1 | 3 | −4 |
| | 1 | 3 | −4 | 0 |

Another factor is $a^2 + 3a - 4$.

$$\therefore a^3 + 2a^2 - 7a + 4 = 0$$
$$(a-1)(a^2 + 3a - 4) = 0$$
$$(a-1)(a+4)(a-1) = 0$$

$$\therefore a - 1 = 0 \quad \text{or} \quad a + 4 = 0$$
$$a = 1 \qquad\qquad a = -4$$

**Check**

For $a = 1$,

| LS | RS |
|---|---|
| $a^3 + 2a^2 - 7a + 4$ | 0 |
| $= 1^3 + 2(1)^2 - 7(1) + 4$ | |
| $= 1 + 2 - 7 + 4$ | |
| $= 0$ | |

LS = RS

For $a = -4$,

| LS | RS |
|---|---|
| $a^3 + 2a^2 - 7a + 4$ | 0 |
| $= (-4)^3 + 2(-4)^2 - 7(-4) + 4$ | |
| $= -64 + 2(16) + 28 + 4$ | |
| $= -64 + 32 + 28 + 4$ | |
| $= 0$ | |

LS = RS

The roots are $-4$ and 1.

**13.** Let $P(x)=x^3-4x^2+x+6$, and use the Integral Zero Theorem to find a factor.

The factors of 6 are $\pm1, \pm2, \pm3$, and $\pm6$.

Try $P(2)$.

$$\begin{aligned} P(x) &= x^3-4x^2+x+6 \\ P(2) &= 2^3-4(2)^2+2+6 \\ &= 8-16+2+6 \\ &= 0 \end{aligned}$$

Therefore, $x-2$ is a factor.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} 2 & 1 & -4 & 1 & 6 \\ & & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Another factor is $x^2-2x-3$.

$$\begin{aligned} &\therefore x^3-4x^2+x+6=0 \\ &(x-2)(x^2-2x-3)=0 \\ &(x-2)(x-3)(x+1)=0 \end{aligned}$$

$$\therefore \quad x-2=0 \quad \text{or} \quad x-3=0 \quad \text{or} \quad x+1=0$$
$$x=2 \qquad\qquad x=3 \qquad\qquad x=-1$$

**Check**

For $x=2$,

| LS | RS |
|---|---|
| $x^3-4x^2+x+6$ | 0 |
| $=2^3-4(2)^2+2+6$ | |
| $=8-16+2+6$ | |
| $=0$ | |

LS = RS

For $x=3$,

| LS | RS |
|---|---|
| $x^3-4x^2+x+6$ | 0 |
| $=3^3-4(3)^2+3+6$ | |
| $=27-36+3+6$ | |
| $=0$ | |

LS = RS

## Section 1: Activity 3 (continued)

For $x=-1$,

| LS | RS |
|---|---|
| $x^3-4x^2+x+6$ | 0 |
| $=(-1)^3-4(-1)^2+(-1)+6$ | |
| $=-1-4-1+6$ | |
| $=0$ | |

LS = RS

Therefore, the roots are $-1$, 2, and 3.

**27.** $(2x+1)(x-1)(x-3)=0$

$\therefore\ 2x+1=0$ or $x-1=0$ or $x-3=0$

$x=-\frac{1}{2}$ $x=1$ $x=3$

The roots are $-\frac{1}{2}$, 1, and 3.

**35.** Let $P(x)=2x^3+9x^2+10x+3$, and use the Integral Zero Theorem to find a factor.

The factors of 3 are $\pm1$ and $\pm3$.

Try $P(-1)$.

$$P(x)=2x^3+9x^2+10x+3$$
$$P(-1)=2(-1)^3+9(-1)^2+10(-1)+3$$
$$=-2+9-10+3$$
$$=0$$

Therefore, $x+1$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| $-1$ | 2 | 9 | 10 | 3 |
| | | $-2$ | $-7$ | $-3$ |
| | 2 | 7 | 3 | 0 |

Another factor is $2x^2+7x+3$.

$$\therefore\ 2x^3+9x^2+10x+3=0$$
$$(x+1)\left(2x^2+7x+3\right)=0$$
$$(x+1)(2x+1)(x+3)=0$$

$$\therefore\ x+1=0 \quad \text{or} \quad 2x+1=0 \quad \text{or} \quad x+3=0$$
$$x=-1 \qquad x=-\frac{1}{2} \qquad x=-3$$

The roots are $-3, -1$, and $-\frac{1}{2}$.

**42.** Let $P(n)=8n^3-26n^2+5n+3$, and use the Integral Zero Theorem to find a factor.

The factors of 3 are $\pm 1$ and $\pm 3$.

Try $P(3)$.

$$P(n)=8n^3-26n^2+5n+3$$
$$P(3)=8(3)^3-26(3)^2+5(3)+3$$
$$=8(27)-26(9)+15+3$$
$$=216-234+18$$
$$=0$$

Therefore, $n-3$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 3 | 8 | −26 | 5 | 3 |
| | | 24 | −6 | −3 |
| | 8 | −2 | −1 | 0 |

Another factor is $8n^2-2n-1$.

$$\therefore\ 8n^3-26n^2+5n+3=0$$
$$(n-3)\left(8n^2-2n-1\right)=0$$
$$(n-3)(4n+1)(2n-1)=0$$

## Section 1: Activity 3 (continued)

$$\therefore\ n-3=0 \quad \text{or} \quad 4n+1=0 \quad \text{or} \quad 2n-1=0$$
$$n=3 \qquad n=-\frac{1}{4} \qquad n=\frac{1}{2}$$

The roots are $-\frac{1}{4}, \frac{1}{2}$, and 3.

**b. Textbook questions 81.a., 82, 83.a., and 86 of "Applications and Problem Solving," p. 220**

**81. a.** Let $P(x)=x^3+kx^2-10x-24$.

If $-2$ is a root of the equation, then $P(-2)=0$.

$$\therefore\ P(x)=x^3+kx^2-10x-24$$
$$P(-2)=(-2)^3+k(-2)^2-10(-2)-24$$
$$0=-8+4k+20-24$$
$$0=4k-12$$
$$4k=12$$
$$k=3$$

$$\therefore\ P(x)=x^3+3x^2-10x-24$$

If $-2$ is a root of the equation, then $-2$ is a zero of $P(x)$. Therefore, $x+2$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| $-2$ | 1 | 3 | $-10$ | $-24$ |
| | | $-2$ | $-2$ | 24 |
| | 1 | 1 | $-12$ | 0 |

Another factor is $x^2+x-12$.

$$\therefore\ x^3+3x^2-10x-24=0$$
$$(x+2)(x^2+x-12)=0$$
$$(x+2)(x+4)(x-3)=0$$

$\therefore\ x+2=0 \quad \text{or} \quad x+4=0 \quad \text{or} \quad x-3=0$

$x=-2 \qquad\qquad x=-4 \qquad\qquad x=3$

The roots are $-4, -2$, and 3.

**82. a.** $x^3-4x^2+kx=0$

$x\left(x^2-4x+k\right)=0$

**Case 1**

If $x^2-4x+k$ is a perfect square, there will be two equal roots.

To determine $k$, divide $-4$ by 2 and square the result.

$$k=\left(\frac{-4}{2}\right)^2$$
$$=(-2)^2$$
$$=4$$

$\therefore\ x^3-4x^2+kx=0$

$x^3-4x^2+4x=0$

$x\left(x^2-4x+4\right)=0$

$x(x-2)(x-2)=0$

$\therefore\ x=0 \quad \text{or} \quad x-2=0$

$x=2$

$x = 2$ is a repeated root.

**Case 2**

If $k=0$,

$x^3-4x^2+kx=0$

$x^3-4x^2=0$

$x^2(x-4)=0$

$\therefore\ x=0 \quad \text{or} \quad x-4=0$

$x=4$

$x = 0$ is a repeated root.

Therefore, $k=0$ and $k=4$.

## Section 1: Activity 3 (continued)

**b.** When $k=0$, the roots are 0 and 4.

When $k=4$, the roots are 0 and 2.

**83. a.** If $x=2$ or $x=-1$ or $x=5$, then $x-2=0$ or $x+1=0$ or $x-5=0$.

$$\therefore (x-2)(x+1)(x-5)=0$$
$$(x-2)\left(x^2-5x+x-5\right)=0$$
$$(x-2)\left(x^2-4x-5\right)=0$$
$$x^3-4x^2-5x-2x^2+8x+10=0$$
$$x^3-6x^2+3x+10=0$$

There are many possible answers because the roots could be repeated. For example, if 2 is repeated, then

$$(x-2)^2(x+1)(x-5)=0$$

This is another equation with $2, -1$, and 5 as solutions.

**86.** Let $P(x)=x^4+2x^3-7x^2-8x+12$, and use the Integral Zero Theorem to find a factor.

The factors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6$, and $\pm12$.

Try $P(1)$.

$$P(x)=x^4+2x^3-7x^2-8x+12$$
$$P(1)=1^4+2(1)^3-7(1)^2-8(1)+12$$
$$=1+2-7-8+12$$
$$=0$$

Therefore, $x-1$ is a factor.

Use synthetic division to find another factor.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | −7 | −8 | 12 |
| | | 1 | 3 | −4 | −12 |
| | 1 | 3 | −4 | −12 | 0 |

Another factor is $x^3+3x^2-4x-12$.

$$\therefore\ x^4+2x^3-7x^2-8x+12=0$$
$$(x-1)\left(x^3+3x^2-4x-12\right)=0$$
$$(x-1)\left[\left(x^3+3x^2\right)+(-4x-12)\right]=0$$
$$(x-1)\left[x^2(x+3)-4(x+3)\right]=0$$
$$(x-1)(x+3)\left(x^2-4\right)=0$$
$$(x-1)(x+3)(x-2)(x+2)=0$$

$\therefore\ x-1=0$ or $x+3=0$ or $x-2=0$ or $x+2=0$

$x=1$ $x=-3$ $x=2$ $x=-2$

The roots are $-3, -2, 1$, and 2.

**3. a. Textbook questions 48, 51, 60, and 65 of "Practice," p. 220**

**48. a.** Let $P(s)=s^3-10s+3$, and use the Integral Zero Theorem to find a factor.

The factors of 3 are $\pm1$ and $\pm3$.

Try $P(3)$.

$$P(s)=s^3-10s+3$$
$$P(3)=3^3-10(3)+3$$
$$=27-30+3$$
$$=0$$

Therefore, $s-3$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 0 | −10 | 3 |
| | | 3 | 9 | −3 |
| | 1 | 3 | −1 | 0 |

Another factor is $s^2+3s-1$.

## Section 1: Activity 3 (continued)

$$\therefore\ s^3 - 10s + 3 = 0$$
$$(s-3)\left(s^2 + 3s - 1\right) = 0$$

$$\therefore\ s - 3 = 0 \quad \text{or} \quad s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$s = 3 \qquad = \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9+4}}{2}$$
$$= \frac{-3 \pm \sqrt{13}}{2}$$

*Note:*
$a = 1$
$b = 2$
$c = -1$

Therefore, the exact roots are 3, $\frac{-3+\sqrt{13}}{2}$, and $\frac{-3-\sqrt{13}}{2}$.

**b.** $\frac{-3+\sqrt{13}}{2} \doteq 0.30$ and $\frac{-3-\sqrt{13}}{2} \doteq -3.30$

Therefore, the approximate roots of the equation $s^3 - 10s + 3 = 0$ are 3, 0.30, and $-3.30$.

**51. a.** $v^3 + 5v^2 = 18$
$v^3 + 5v^2 - 18 = 0$

Let $P(v) = v^3 + 5v^2 - 18$, and use the Integral Zero Theorem to find a factor.

The factors of $-18$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$, and $\pm 18$.

Try $P(-3)$.

$$P(v) = v^3 + 5v^2 - 18$$
$$P(-3) = (-3)^3 + 5(-3)^2 - 18$$
$$= -27 + 45 - 18$$
$$= 0$$

Therefore, $v + 3$ is a factor.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} -3 & 1 & 5 & 0 & -18 \\ & & -3 & -6 & 18 \\ \hline & 1 & 2 & -6 & 0 \end{array}$$

Another factor is $v^2+2v-6=0$.

$$\therefore\ v^3+5v^2-18=0$$
$$(v+3)\left(v^2+2v-6\right)=0$$

$$\therefore\ v+3=0 \quad \text{or} \quad v=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$v=-3$$
$$=\frac{-2\pm\sqrt{2^2-4(1)(-6)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{4+24}}{2}$$
$$=\frac{-2\pm\sqrt{28}}{2}$$
$$=\frac{-2\pm2\sqrt{7}}{2}$$
$$=\frac{\cancel{2}\left(-1\pm\sqrt{7}\right)}{\cancel{2}}$$
$$=-1\pm\sqrt{7}$$

*Note:*
$a=1$
$b=2$
$c=-6$

Therefore, the exact roots are $-3, -1+\sqrt{7}$, and $-1-\sqrt{7}$.

**b.** $-1+\sqrt{7}=1.65$ and $-1-\sqrt{7}=-3.65$

Therefore, the approximate roots of the equation $v^3+5v^2=18$ are $-3, 1.65$, and $-3.65$.

## Section 1: Activity 3 (continued)

**60.** Let $P(x)=x^3+x+2$, and use the Integral Zero Theorem to find a factor.

The factors of 2 are $\pm 1$ and $\pm 2$.

Try $P(-1)$.

$$\begin{aligned} P(x)&=x^3+x+2\\ P(-1)&=(-1)^3+(-1)+2\\ &=-1-1+2\\ &=0 \end{aligned}$$

Therefore, $x+1$ is a factor.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} -1 & 1 & 0 & 1 & 2\\ & & -1 & 1 & -2\\ \hline & 1 & -1 & 2 & 0 \end{array}$$

Another factor is $x^2-x+2=0$.

$$\therefore\ x^3+x+2=0$$

$$(x+1)\left(x^2-x+2\right)=0$$

$$\therefore\ x+1=0 \quad \text{or} \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=-1$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(2)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1-8}}{2}$$

$$=\frac{1\pm\sqrt{-7}}{2}$$

$$=\frac{1\pm i\sqrt{7}}{2}$$

*Note:*
$a=1$
$b=-1$
$c=2$

The exact roots are $-1, \frac{1+i\sqrt{7}}{2}$, and $\frac{1-i\sqrt{7}}{2}$.

**65.** Let $P(n)=3n^3+9n^2+10n+4=0$, and use the Integral Zero Theorem to find a factor.

The factors of 4 are $\pm 1, \pm 2$, and $\pm 4$.

Try $P(-1)$.

$$\begin{aligned} P(n) &= 3n^3+9n^2+10n+4 \\ P(-1) &= 3(-1)^3+9(-1)^2+10(-1)+4 \\ &= -3+9-10+4 \\ &= 0 \end{aligned}$$

Therefore, $n+1$ is a factor.

Use synthetic division to find another factor.

$$\begin{array}{c|cccc} -1 & 3 & 9 & 10 & 4 \\ & & -3 & -6 & -4 \\ \hline & 3 & 6 & 4 & 0 \end{array}$$

Another factor is $3n^2+6n+4$.

$$\therefore\ 3n^3+9n^2+10n+4=0$$
$$(n+1)\left(3n^2+6n+4\right)=0$$

## Section 1: Activity 3 (continued)

$\therefore\ n+1=0 \quad$ or $\quad n=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$n=-1$

$=\dfrac{-6\pm\sqrt{6^2-4(3)(4)}}{2(3)}$

$=\dfrac{-6\pm\sqrt{36-48}}{6}$

$=\dfrac{-6\pm\sqrt{-12}}{6}$

$=\dfrac{-6\pm 2i\sqrt{3}}{6}$

$=\dfrac{\overset{1}{\cancel{2}}\left(-3\pm i\sqrt{3}\right)}{\underset{3}{\cancel{6}}}$

$=\dfrac{-3\pm i\sqrt{3}}{3}$

*Note:*
$a=3$
$b=6$
$c=4$

The exact roots are $-1, \frac{-3+i\sqrt{3}}{3}$, and $\frac{-3-i\sqrt{3}}{3}$.

**b. Textbook questions 75, 80, 93, 100, and 104 of "Applications and Problem Solving," pp. 220 and 221**

**75.** $3x^3+8x^2=1$

$3x^3+8x^2-1=0$

Let $P(x)=3x^3+8x^2-1$, and use the Rational Zero Theorem to find a factor.

If $3x^3+8x^2-1$ has a factor $ax-b$, then $b$ is a factor of the constant term, which is $-1$, and $a$ is a factor of the coefficient of $x^3$, which is 3.

The factors of $-1$ are $\pm 1$; the factors of 3 are $\pm 1$ and $\pm 3$. Therefore, the possible ratios of $\frac{b}{a}$ are $\pm 1$ and $\pm\frac{1}{3}$.

Try $P\left(\frac{1}{3}\right)$.

$$P(x)=3x^3+8x^2-1$$
$$P\left(\frac{1}{3}\right)=3\left(\frac{1}{3}\right)^3+8\left(\frac{1}{3}\right)^2-1$$
$$=\cancel{3}^{1}\left(\frac{1}{\cancel{27}\,9}\right)+8\left(\frac{1}{9}\right)-1$$
$$=\frac{1}{9}+\frac{8}{9}-1$$
$$=0$$

Therefore, $x-\frac{1}{3}$ or $3x-1$ is a factor.

Use synthetic division to find another factor.

| $\frac{1}{3}$ | 3 | 8 | 0 | −1 |
|---|---|---|---|---|
| | | 1 | 3 | 1 |
| | 3 | 9 | 3 | 0 |

Another factor is $3x^2+9x+3$.

$$\therefore\ 3x^3+8x^2-1=0$$
$$\left(x-\frac{1}{3}\right)\left(3x^2+9x+3\right)=0$$
$$\left(x-\frac{1}{3}\right)(3)\left(x^2+3x+1\right)=0$$
$$(3x-1)\left(x^2+3x+1\right)=0$$

## Section 1: Activity 3 (continued)

$\therefore\ 3x-1=0$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{1}{3}$ $=\dfrac{-3\pm\sqrt{3^2-4(1)(1)}}{2(1)}$

$=\dfrac{-3\pm\sqrt{9-4}}{2}$

$=\dfrac{-3\pm\sqrt{5}}{2}$

*Note:*
$a=1$
$b=3$
$c=1$

The roots are $\frac{1}{3}$, $\frac{-3+\sqrt{5}}{2}$, and $\frac{-3-\sqrt{5}}{2}$.

**80.** $\dfrac{2}{x-2}-\dfrac{1}{x-1}=x$

*Note:* Multiply by the LCD, $(x-2)(x-1)$.

$2(x-1)-(x-2)=x(x-2)(x-1)$

$2x-2-x+2=x\left(x^2-x-2x+2\right)$

$x=x\left(x^2-3x+2\right)$

$x=x^3-3x^2+2x$

$0=x^3-3x^2+x$

$x^3-3x^2+x=0$

$x\left(x^2-3x+1\right)=0$

$\therefore\ x=0$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$

$=\dfrac{3\pm\sqrt{9-4}}{2}$

$=\dfrac{3\pm\sqrt{5}}{2}$

*Note:*
$a=1$
$b=-3$
$c=1$

The roots are 0, $\frac{3+\sqrt{5}}{2}$, and $\frac{3-\sqrt{5}}{2}$.

**93.** $V = \ell \times w \times h$

$$42 = (x+3)(x-2)(x-1)$$

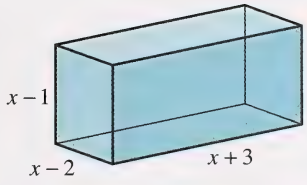
$$42 = (x+3)(x^2 - x - 2x + 2)$$

$$42 = (x+3)(x^2 - 3x + 2)$$

$$42 = x^3 - 3x^2 + 2x + 3x^2 - 9x + 6$$

$$42 = x^3 - 7x + 6$$

$$0 = x^3 - 7x - 36$$




Let $P(x) = x^3 - 7x - 36$, and use the Integral Zero Theorem to find a factor.

The factors of $-36$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18,$ and $\pm 36$.

Try $P(4)$.

$$P(x) = x^3 - 7x - 36$$

$$P(4) = 4^3 - 7(4) - 36$$

$$= 64 - 28 - 36$$

$$= 0$$

Therefore, $x - 4$ is a factor.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 0 | −7 | −36 |
| | | 4 | 16 | 36 |
| | 1 | 4 | 9 | 0 |

Another factor is $x^2 + 4x + 9$.

$$\therefore x^3 - 7x - 36 = 0$$

$$(x-4)(x^2 + 4x + 9) = 0$$

## Section 1: Activity 3 (continued)

$\therefore\ x-4=0$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=4$

$=\dfrac{-4\pm\sqrt{4^2-4(1)(9)}}{2(1)}$

$=\dfrac{-4\pm\sqrt{16-36}}{2}$

$=\dfrac{-4\pm\sqrt{-20}}{2}$

$=\dfrac{-4\pm 2i\sqrt{5}}{2}$

$=\dfrac{\cancel{2}\left(-2\pm i\sqrt{5}\right)}{\cancel{2}}$

$=-2\pm i\sqrt{5}$

*Note:*
$a=1$
$b=4$
$c=9$

Because the dimensions must be non-negative real numbers, $x=4$.

$\therefore\ x+3=4+3$ $\quad x-2=4-2$ $\quad x-1=4-1$

$=7$ $\quad =2$ $\quad =3$

The rectangular solid is 7 cm × 2 cm × 3 cm.

**100.** Answers may vary. A sample answer is given.

If $x=5$, then $x-5=0$.
If $x=2$, then $x-2=0$.
If $x=-3$, then $x+3=0$.

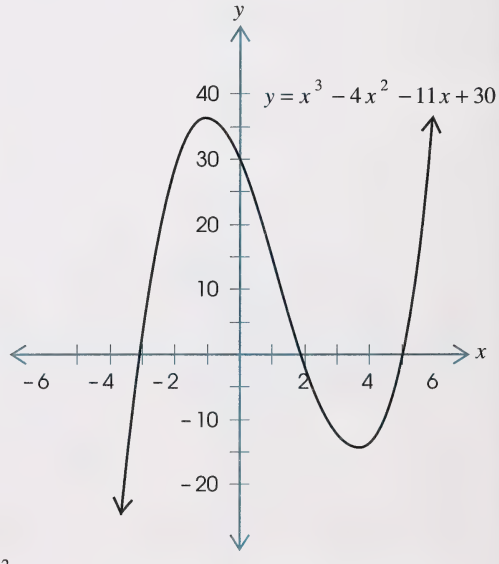
$\therefore\ (x-5)(x-2)(x+3)=0$

$(x-5)\left(x^2+3x-2x-6\right)=0$

$(x-5)\left(x^2+x-6\right)=0$

$x^3+x^2-6x-5x^2-5x+30=0$

$x^3-4x^2-11x+30=0$




A corresponding cubic function is $y=x^3-4x^2-11x+30$. Its graph will cross the $x$-axis at $(5,0),(2,0)$, and $(-3,0)$.

There are many possible graphs. You can use any cubic function that intersects the $x$-axis at $(-3,0),(2,0)$, and $(5,0)$.

Another cubic function is $y=-1(x+3)(x-2)(x-5)$. A third is $y=2(x+3)(x-2)(x-5)$.

**104. a.** If $x=\sqrt{5}$, then $x-\sqrt{5}=0$.
If $x=-\sqrt{5}$, then $x+\sqrt{5}=0$.
If $x=-1$, then $x+1=0$.

$$\therefore \left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)(x+1)=0$$
$$\left(x^2-5\right)(x+1)=0$$
$$x^3+x^2-5x-5=0$$

A corresponding cubic function is $y=x^3+x^2-5x-5$.

When $x=0$, $y=-5$. Therefore, the $y$-intercept is $-5$.

Thus, a cubic function with $x$-intercepts of $\sqrt{5}$, $-\sqrt{5}$, and $-1$ and a $y$-intercept of $-5$ is $y=x^3+x^2-5x-5$.

**b.** A cubic function with the same $x$-intercepts and a $y$-intercept of $-10$ is $y=2x^3+2x^2-10x-10$. (When $x=0$, $y=-10$.) Make sure this function has the same zeros.

$$y=2x^3+2x^2-10x-10$$
$$y=2\left(x^3+x^2-5x-5\right)$$
$$y=2\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)(x+1)$$

Thus, the graph of this function intersects the $x$-axis at $\left(\sqrt{5},0\right),\left(-\sqrt{5},0\right)$, and $(1,0)$ and the $y$-axis at $(0,-10)$.

## Section 1: Activity 3 (continued)

**4. Textbook question "LOGIC POWER," p. 221**

Answers will vary. Two sample answers are given.

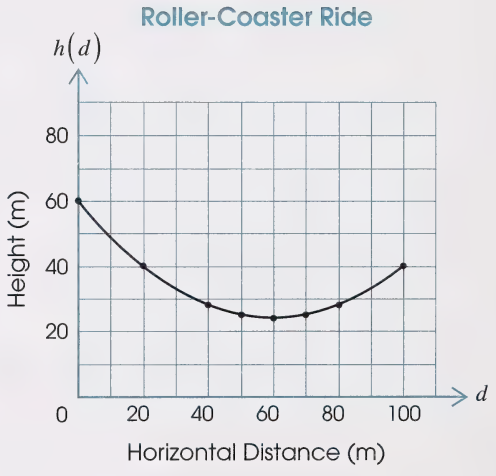
| | | | | |
|---|---|---|---|---|
| | 1 | 3 | | |
| 12 | 5 | 7 | 9 | |
| 10 | 2 | 11 | 4 | |
| | 6 | 8 | | |

| | | | | |
|---|---|---|---|---|
| | 11 | 9 | | |
| 5 | 1 | 4 | 12 | |
| 3 | 8 | 10 | 7 | |
| | 6 | 2 | | |

## Section 1: Activity 4

**1. a. Textbook question "Explore: Draw a Graph," p. 274**

| *d* (m) | *h*(*d*) (m) |
|---|---|
| 0 | 60 |
| 20 | 40 |
| 40 | 28 |
| 50 | 25 |
| 60 | 24 |
| 70 | 25 |
| 80 | 28 |
| 100 | 40 |



**b. Textbook questions 1 to 5 of "Inquire," p. 274**

**1.** At its lowest point, the train is 24 m above the ground.

**2.** **a.** The domain is $0 \le d \le 100$.
**b.** The range is $24 \le h(d) \le 60$.

**3.** The function is of degree 2.

**4.** The function has no real zeros; its graph does not cross the $x$-axis.

**5.** The graph is continuous within its domain; there are no breaks or gaps in the graph.

**2. Textbook questions 1 to 5 of "Practice," p. 282**

**1.** The function $y = 2x^5 - 3x - 4$ is a polynomial function because the exponents of the powers of $x$ are whole numbers. The funcion is of degree 5.

**2.** The function $y = 2x^2 - 4x^{-1} + 7$ is not a polynomial function because the exponent $-1$ is not a whole number.

**3.** The function $y = 7 - 4x + \frac{1}{x^2}$ is not a polynomial function. If you write the function as $y = 7 - 4x + x^{-2}$, you will notice that the exponent $-2$ is not a whole number.

**4.** The function $y = 3x^4 + 0.005x^3 - \sqrt{6}$ is a polynomial function because the exponents of the powers of $x$ are whole numbers. The function is of degree 4.

**5.** The function $y = \sqrt{x} + x^5 - 7x^6$ is not a polynomial function. If you write the function as $y = x^{\frac{1}{2}} + x^5 - 7x^6$, you will notice that the exponent $\frac{1}{2}$ is not a whole number.

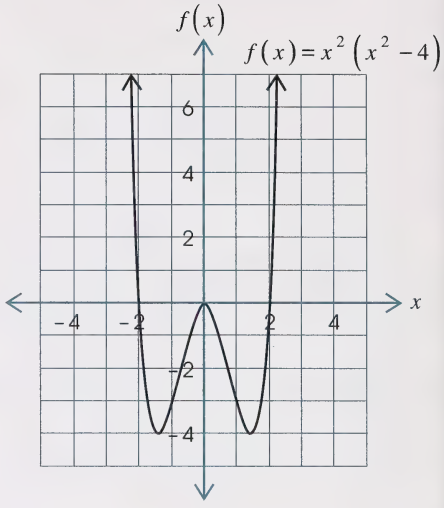
**3. Textbook questions 10, 17, 23, 29, and 30 of "Practice," pp. 282 and 283**

**10.** The domain is the set of real numbers, and the range is the set of real numbers.

From the graph, there is a local minimum at $(0, 2)$ and a local maximum at $(2, 6)$.

The $y$-intercept is 2.

## Section 1: Activity 4 (continued)




**17. a.** The domain is the set of real numbers, and the range is $f(x) \geq -4$. **Note:** $f(x)$, $g(x)$, $h(x)$, and so on can be used to replace $y$.

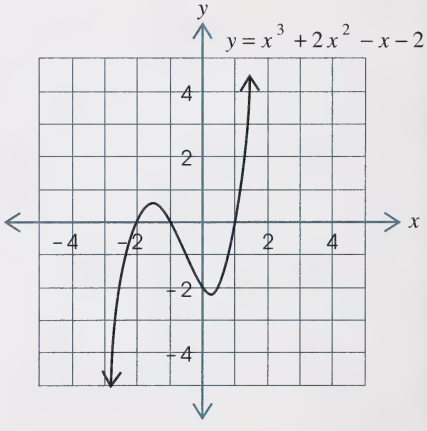
**b.** The zeros are $-2, 0$, and 2.

**c.** The $y$-intercept is 0.

**d.** From the graph, $f(x)$ has a local minimum at about $(-1.4, -4)$ and $(1.4, -4)$ and a local maximum at $(0, 0)$.

**e.** The graph is symmetric about the $y$-axis.

**f.** The left-most $y$-values are positive, and the right-most $y$-values are positive.




**23. a.** The domain is the set of real numbers, and the range is the set of real numbers.

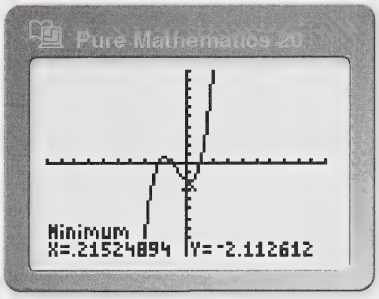
**b.** From the graph, the zeros appear to be $-2, -1$, and 1. To confirm these values algebraically, factor the polynomial.

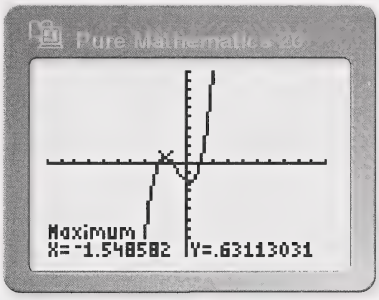
$$\begin{aligned} x^3 + 2x^2 - x - 2 &= \left(x^3 + 2x^2\right) + (-x - 2) \\ &= x^2(x+2) - 1(x+2) \\ &= (x+2)\left(x^2 - 1\right) \\ &= (x+2)(x+1)(x-1) \end{aligned}$$

The zeros are $-2, -1$, and 1.

**c.** The $y$-intercept is $-2$.

**d.** Use the Minimum and Maximum features on your graphing calculator to locate the local minimum and local maximum.



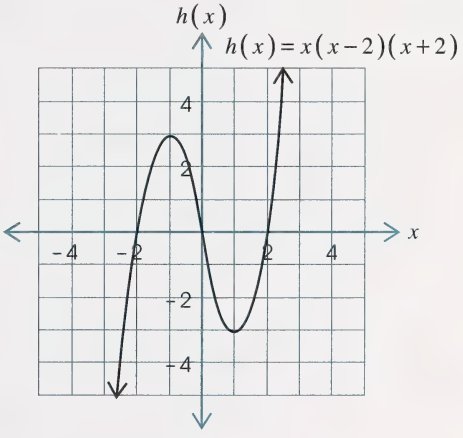




The graph has a local minimum at about $(0.2, -2.1)$ and a local maximum at about $(-1.5, 0.6)$.

**e.** The graph does not contain any symmetry.

**f.** The left-most $y$-values are negative, and the right-most $y$-values are positive.

**29. a.** The domain is the set of real numbers, and the range is the set of real numbers.




**b.** The real zeros are $-2, 0$, and 2.

**c.** The $y$-intercept is 0.

**d.** Use the Minimum and Maximum features on your graphing calculator. The local minimum is about $(1.2, -3.1)$, and the local maximum is about $(-1.2, 3.1)$.

**e.** The graph is symmetric about the origin.

**f.** The left-most $y$-values are negative, and the right-most $y$-values are positive.

## Section 1: Activity 4 (continued)

**30. a.** The domain is the set of real numbers, and the range is the set of real numbers.

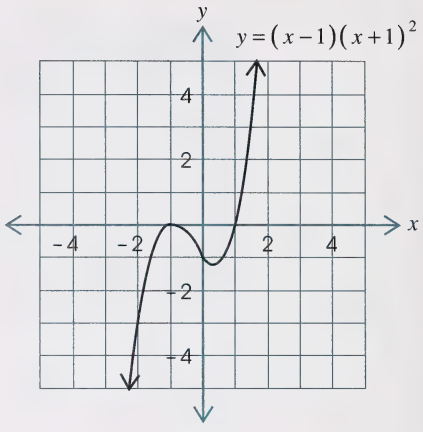
**b.** The zeros are $-1$ and $1$.

**c.** The $y$-intercept is $-1$.

**d.** Use the Minimum and Maximum features on your graphing calculator. The local minimum is about $(0.3, -1.2)$. The local maximum is $(-1, 0)$.

**e.** The graph does not contain any symmetry.

**f.** The left-most $y$-values are negative, and the right-most $y$-values are positive.




**4. a. Textbook questions 33, 41, 46, 49, and 54 of "Practice," p. 283**

**33. a.** The domain is the set of real numbers, and the range is the set of real numbers.

**b.** The zeros are $-1, 2$, and $4$.

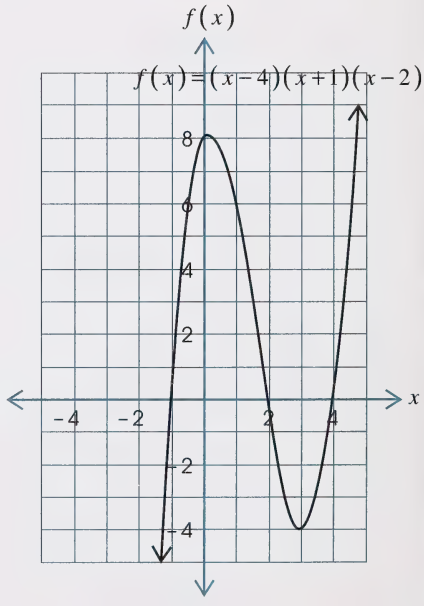
**c.** The $y$-intercept is $8$.

**d.** The function $f(x) > 0$ when $-1 < x < 2$ and when $x > 4$.

The function $f(x) \leq 0$ when $x \leq -1$ and when $2 \leq x \leq 4$.

**e.** The graph does not contain any symmetry.

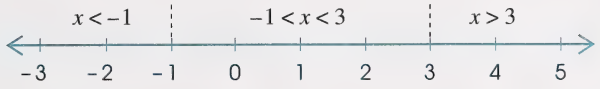
**f.** The left-most $y$-values are negative and the right-most $y$-values are positive.



**41.** Find the zeros of the function $f(x)=(x+1)(x-3)$.

The zeros are $-1$ and 3.

These values divide the $x$-axis into three test intervals.




Determine the signs of the factors of $f(x)$ in each interval; then determine the sign of $f(x)$.

| Interval | $x+1$ | $x-3$ | $f(x)$ |
|---|---|---|---|
| $x<-1$ | − | − | + |
| $-1<x<3$ | + | − | − |
| $x>3$ | + | + | + |

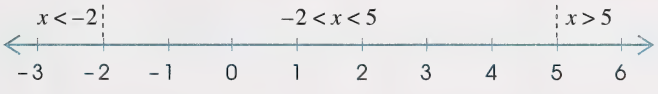
From the table, $(x+1)(x-3)>0$ when $x<-1$ or when $x>3$.

**46.** $x^2-3x\le 10$

$x^2-3x-10\le 0$

$(x-5)(x+2)\le 0$

The zeros of $f(x)=(x-5)(x+2)$ are $-2$ and 5. The zeros divide the $x$-axis into three test intervals.




Determine the signs of the factors of $f(x)$ and of $f(x)$ itself.

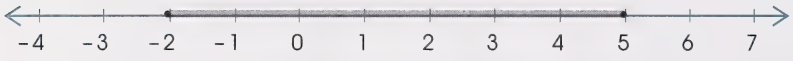
| Interval | $x-5$ | $x+2$ | $f(x)$ |
|---|---|---|---|
| $x<-2$ | − | − | + |
| $-2<x<5$ | − | + | − |
| $x>5$ | + | + | + |

## Section 1: Activity 4 (continued)

Therefore, $(x-5)(x+2)\leq 0$ when $-2\leq x\leq 5$.

The solution, graphed on a number line, is as follows:

Because the original inequality contained the symbol ≤, include −2 and 5 in your solution.




**49.** $(x-1)(x+1)(x+2)<0$

The zeros of $f(x)=(x-1)(x+1)(x+2)$ are $-2, -1$, and 1. The zeros divide the $x$-axis into four test intervals.

Determine the signs of the factors of $f(x)$ and of $f(x)$ itself.

| Interval | $x-1$ | $x+1$ | $x+2$ | $f(x)$ |
|---|---|---|---|---|
| $x<-2$ | − | − | − | − |
| $-2<x<-1$ | − | − | + | + |
| $-1<x<1$ | − | + | + | − |
| $x>1$ | + | + | + | + |

Therefore, $(x-1)(x+1)(x+2)<0$ when $x<-2$ or when $-1<x<1$.

**54.**
$$x^3-2x^2-x+2\geq 0$$
$$(x^3-2x^2)+(-x+2)\geq 0$$
$$x^2(x-2)-1(x-2)\geq 0$$
$$(x-2)(x^2-1)\geq 0$$
$$(x-2)(x-1)(x+1)\geq 0$$

The zeros of $f(x)=(x-2)(x-1)(x+1)$ are $-1$, 1, and 2. The zeros divide the $x$-axis into four test intervals.

Determine the signs of the factors of $f(x)$ and of $f(x)$ itself.

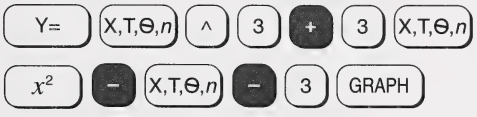
| Interval | $x-2$ | $x-1$ | $x+1$ | $f(x)$ |
|---|---|---|---|---|
| $x<-1$ | $-$ | $-$ | $-$ | $-$ |
| $-1<x<1$ | $-$ | $-$ | $+$ | $+$ |
| $1<x<2$ | $-$ | $+$ | $+$ | $-$ |
| $x>2$ | $+$ | $+$ | $+$ | $+$ |

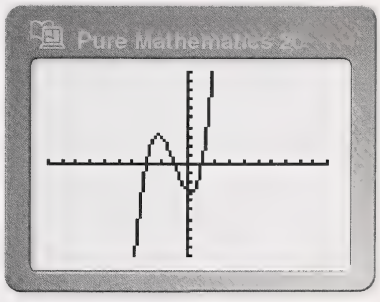
Therefore, $x^3-2x^2-x+2\geq 0$ when $-1\leq x\leq 1$ or when $x\geq 2$.

**b. Textbook questions 57.c., 62, 64, and 70 of "Applications and Problem Solving," pp. 283 to 285**

**57. c.** $x^3+3x^2-x-3<0$

Using the standard window settings, graph the corresponding cubic function, $f(x)=x^3+3x^2-x-3$, on your graphing calculator.



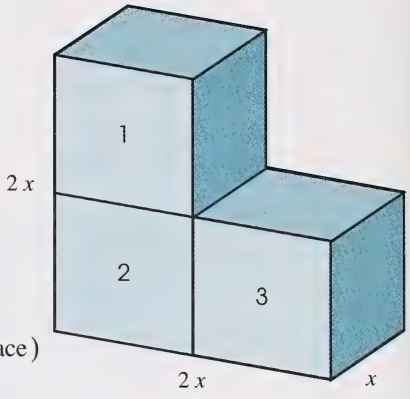

The $x$-intercepts of the graph are $-3$, $-1$, and 1.

The solutions of $x^3+3x^2-x-3<0$ are those values of $x$ for which the graph of $f(x)$ lies below the $x$-axis.

Therefore, the solution is $x<-3$ or $-1<x<1$.

## Section 1: Activity 4 (continued)

**62. a.** Let $A(x)$ be the surface area of the solid.

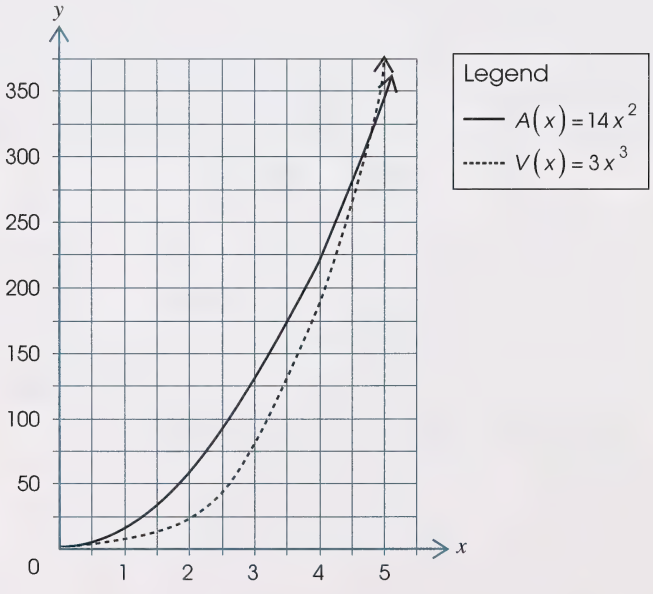



Each face of a cube has a surface area of $x^2$; therefore, each separate cube has a surface area of $6x^2$. However, when the cubes are put together to form the solid, four faces of the cube are concealed: the bottom of cube 1, the top of cube 2, the right face of cube 2, and the left face of cube 3.

$$\therefore A(x) = 3(\text{surface area of 1 cube}) - 4(\text{area of 1 face})$$
$$= 3(6x^2) - 4(x^2)$$
$$= 18x^2 - 4x^2$$
$$= 14x^2, \text{ where } x \geq 0$$

Let $V(x)$ be the volume of the solid.

$$\therefore V(x) = 3(\text{volume of 1 cube})$$
$$= 3x^3, \text{ where } x \geq 0$$



**b.** $A(x) = V(x)$

$14x^2 = 3x^3$

$0 = 3x^3 - 14x^2$

$3x^3 - 14x^2 = 0$

$x^2(3x - 14) = 0$

$\therefore x^2 = 0$ or $3x - 14 = 0$

$x = 0$ $3x = 14$

$x = \frac{14}{3}$ or $4\frac{2}{3}$

The surface area and the volume will be equal when the edge length, $x$, of each cube is 0 or $4\frac{2}{3}$.

**c.** $A(x) > V(x)$ when $0 < x < \frac{14}{3}$.

$A(x) < V(x)$ when $x > \frac{14}{3}$.

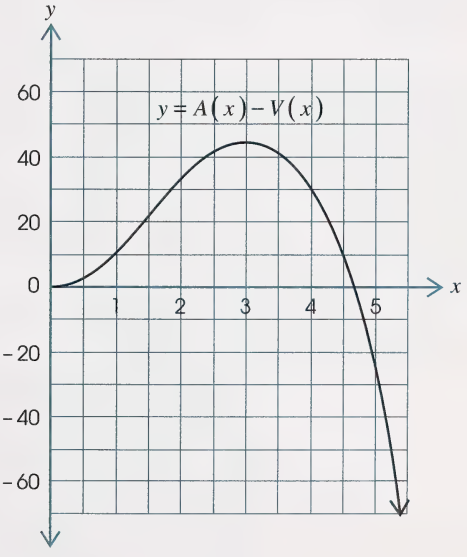
**d.** $y = A(x) - V(x)$

$= 14x^2 - 3x^3$

When $A(x) - V(x) > 0$ (that is, when the graph lies above the $x$-axis), the surface area is greater than the volume.

When $A(x) - V(x) < 0$ (that is, when the graph lies below the $x$-axis), the surface area is less than the volume.

The graph crosses the $x$-axis at $x = 4\frac{2}{3}$, the value when $A(x) = V(x)$.



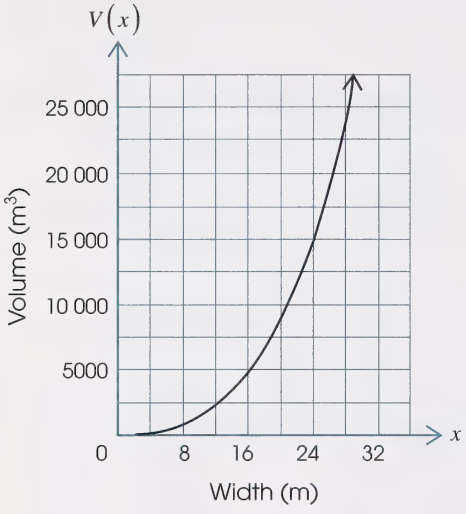
## Section 1: Activity 4 (continued)

**64. a.** Let $x$ represent the width.

$$\therefore V(x) = \ell \times w \times h$$
$$= (x+5)(x)(x-2)$$

**b.** The domain of the function is $x \geq 2$ because height is non-negative.

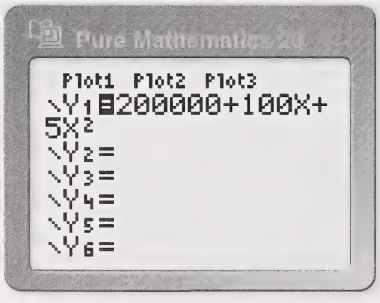
**c.**

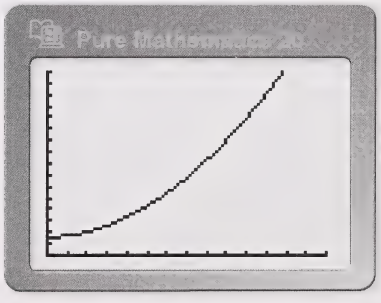



**d.** The volume of air in a warehouse with a width of 20 m is about 9000 $m^3$.

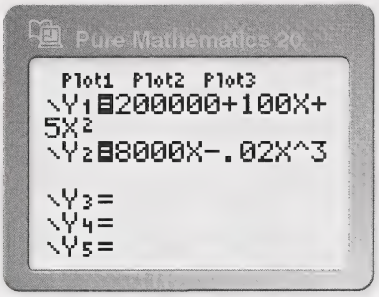
**e.** The maximum width of a warehouse in which the system could be used is about 28 m.

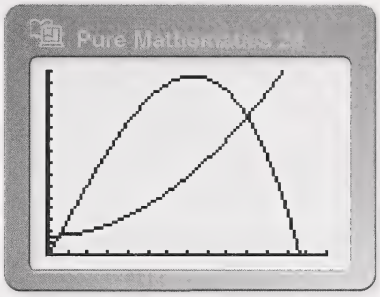
**70. a.** First, graph the cost function, $C(x) = 200\,000 + 100x + 5x^2$.



On the same set of axes, graph the revenue function, $R(x) = 8000x - 0.02x^3$

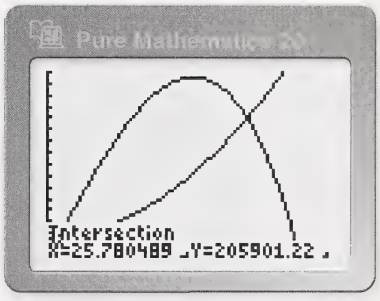



**b.** In order for the company to make a profit, revenue must be greater than cost. This occurs when the revenue graph lies above the cost graph.

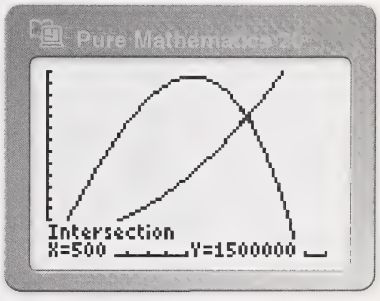
Use the Intersect feature from the CALCULATE menu to find the minimum number of trailers.

Because the intersection of these two graphs is the point when revenue equals cost, the company needs to sell a minimum of 26 trailers in order to make a profit.




**c.** Use the Intersect feature from the CALCULATE menu to find the maximum number of trailers.

Because the intersection of these two graphs is the point when revenue equals cost, the company needs to sell a maximum of 499 trailers in order to make a profit.



## Section 1: Activity 4 (continued)

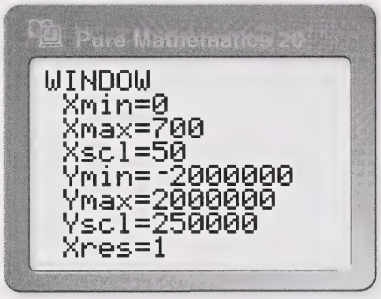
**d.** $P(x) = R(x) - C(x)$

$= \left(8000x - 0.02x^3\right) - \left(200\,000 + 100x + 5x^2\right)$

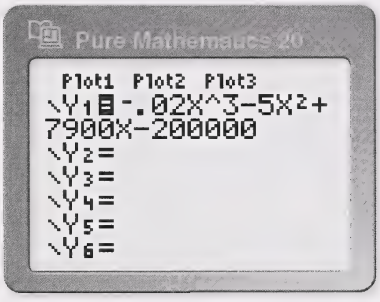
$= 8000x - 0.02x^3 - 200\,000 - 100x - 5x^2$

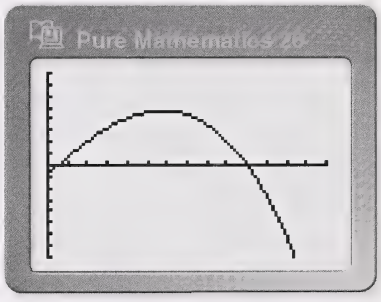
$= -0.02x^3 - 5x^2 + 7900x - 200\,000$

**e.** Before graphing the profit function, set the window setting as follows:

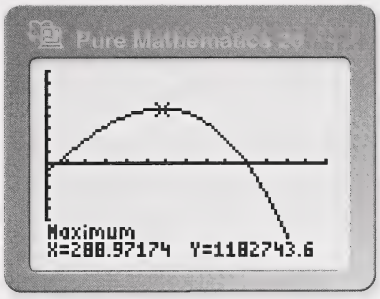



Now, graph the function $P(x) = -0.02x^3 - 5x^2 + 7900x - 200\,000$.



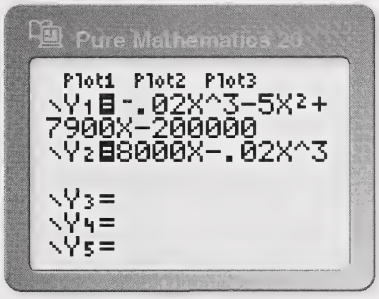



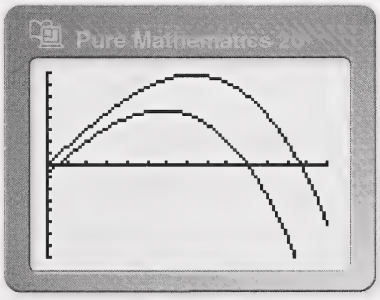
Use the Maximum feature from the CALCULATE menu to find the maximum profit.




A maximum profit of about $1 180 000 occurs when 289 trailers are manufactured.

**f.** On the same set of axes, graph the profit function and the revenue function.




Because the graphs never intersect, this shows that profits will never be greater than revenues.

**g.** The company cannot make a profit if costs are greater than revenues. According to the equation $P(x) = R(x) - C(x)$, profits show a negative value (or a loss) if costs are greater.

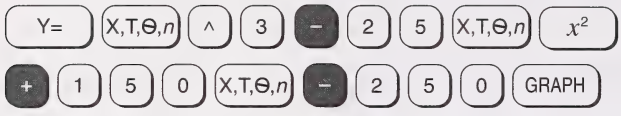
The company can make a profit if costs are greater than profits. Profit is the amount of money the company receives after paying all expenses involved in production. So, if a trailer is sold for $100 and the cost of producing the trailer is $90, then the profit made is $10, which is less than the cost. Even though the company made only a $10 profit, it still made a profit.

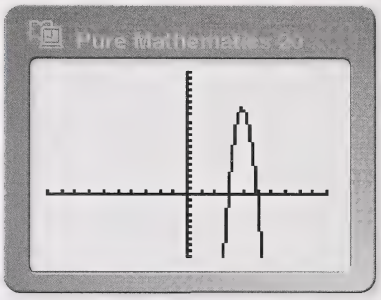
## Section 1: Activity 4 (continued)

**5.** $4x^3 - 100x^2 + 600x - 1000 \geq 0$

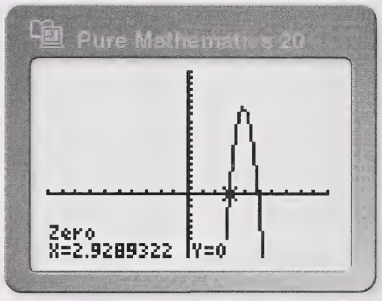
$x^3 - 25x^2 + 150x - 250 \geq 0$ ← Divide both sides by 4.

Graph the corresponding function, $y = x^3 - 25x^2 + 150x - 250$, on your graphing calculator. Use the standard window settings, but change Ymax to 20.



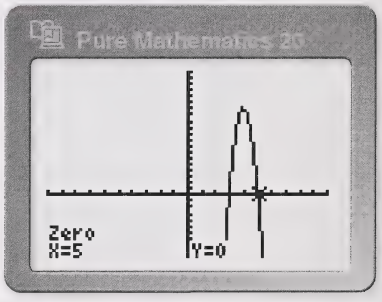

Use the Zero feature from the CALCULATE menu to determine the approximate $x$-intercepts.




Use the Factor Theorem to check the $x$-intercepts.

Try $f(5)$.

$$f(x) = x^3 - 25x^2 + 150x - 250$$
$$f(5) = 5^3 - 25(5)^2 + 150(5) - 250$$
$$= 125 - 625 + 750 - 250$$
$$= 0$$




Therefore, $x - 5$ is a factor of $f(x)$.

Use synthetic division to find another factor.

| | | | | |
|---|---|---|---|---|
| 5 | 1 | −25 | 150 | −250 |
| | | 5 | −100 | 250 |
| | 1 | −20 | 50 | 0 |

Another factor is $x^2 - 20x + 50$.

$$\therefore f(x) = x^3 - 25x^2 + 150x - 250$$
$$= (x - 5)(x^2 - 20x + 50)$$

$\therefore x-5=0$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=5$

$$=\frac{-(-20)\pm\sqrt{(-20)^2-4(1)(50)}}{2(1)}$$

$$=\frac{20\pm\sqrt{400-200}}{2}$$

$$=\frac{20\pm\sqrt{200}}{2}$$

$$=\frac{20\pm10\sqrt{2}}{2}$$

$$=10\pm5\sqrt{2}$$

*Note:*
$a=1$
$b=-20$
$c=50$

$10+5\sqrt{2}\doteq17.07$ and $10-5\sqrt{2}\doteq2.93$

However, the largest value of $x$ is 10 because the original sheet was 20 cm in width. The largest square you could cut from each corner would be $\frac{1}{2}(20\text{ cm})$ or 10 cm on a side.

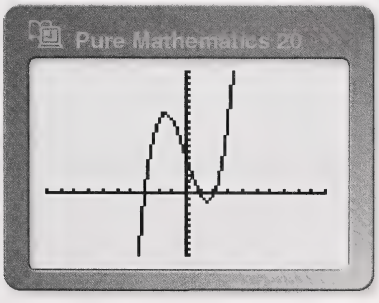
Therefore, the $x$-intercepts (zeros) are about 2.93 and 5. The graph lies above the $x$-axis between these values. Thus, $4x^3-100x^2+600x-1000\geq0$ when $10-5\sqrt{2}<x<5$ or approximately $2.93<x<5$.

## Section 1: Follow-up Activities

### Extra Help

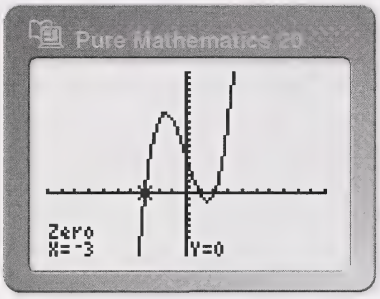
**1.** $f(x)=x^3-7x+6$

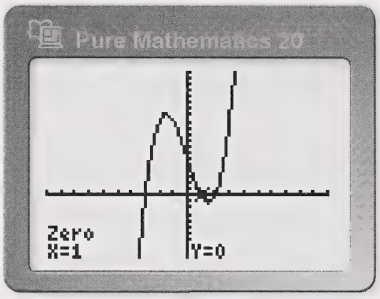
**a.** Use the standard window settings, but change Ymax to 20.

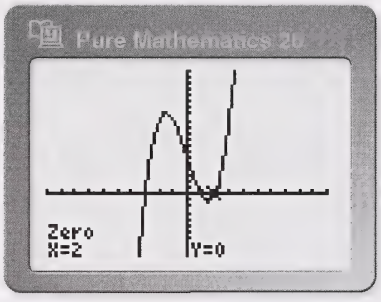


## Extra Help (continued)

**b.** The graph appears to cross the $x$-axis at $x = -3$, 1, and 2. Confirm these values using the Zero feature from the CALCULATE menu.

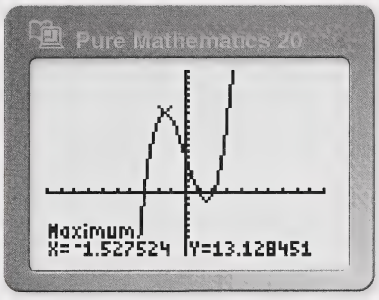



**c.** The $y$-intercept is $(0, 6)$.

**d.** Use the Maximum feature from the CALCULATE menu to locate the relative maximum point; then use the Minimum feature from the CALCULATE menu to locate the relative minimum point.

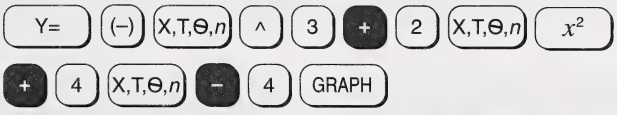


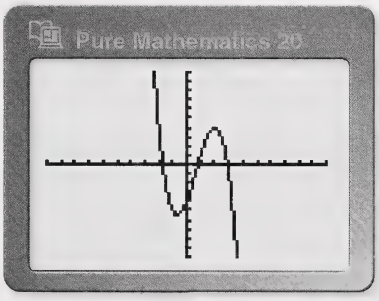
Therefore, a relative maximum point occurs at about $(-1.53, 13.13)$ and a relative minimum point occurs at about $(1.53, -1.13)$.

**e.** The left-most $y$-values are negative. The right-most $y$-values are positive.

**2.** $f(x) = -x^3 + 2x^2 + 4x - 4$

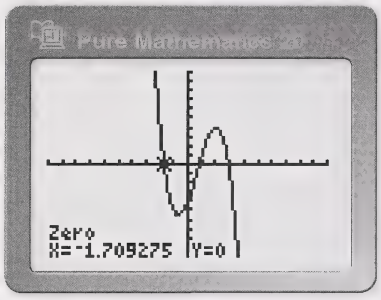
**a.** Use the standard window settings.

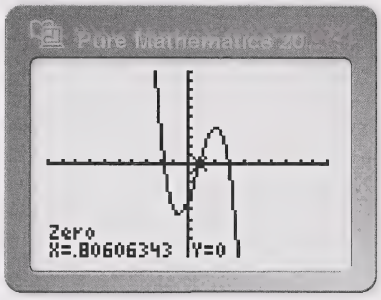


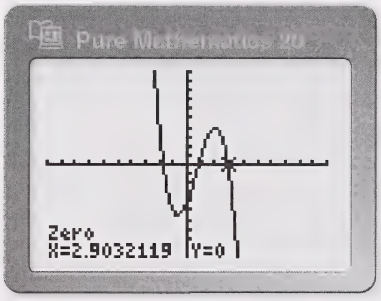



## Extra Help (continued)

**b.** Use the Zero feature from the CALCULATE menu to approximate the zeros ($x$-intercepts).



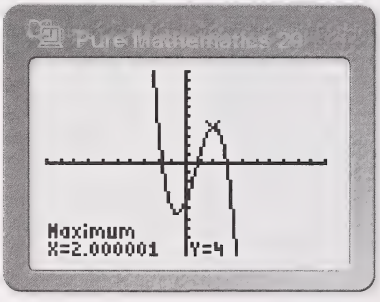



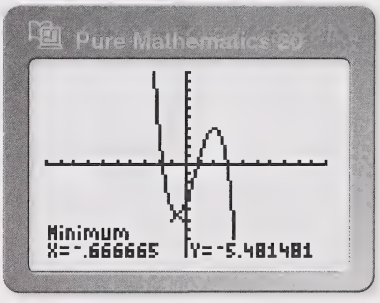




The zeros are approximately $-1.71$, $0.81$, and $2.90$.

**c.** The $y$-intercept is $-4$.

**d.** Use the Maximum feature from the CALCULATE menu to find the relative maximum point; then use the Minimum feature from the CALCULATE menu to find the relative minimum point.



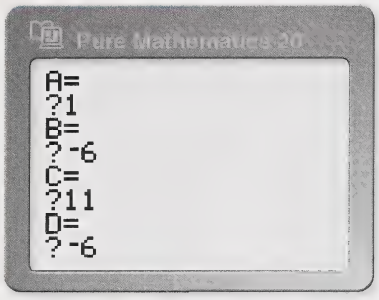



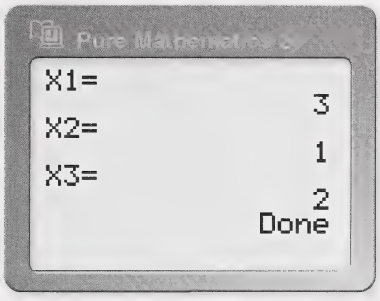

Therefore, the relative maximum point occurs at about $(2, 4)$, and the relative minimum point occurs at about $(-0.67, -5.48)$.

**e.** The left-most $y$-values are positive, and the right-most $x$-values are negative.

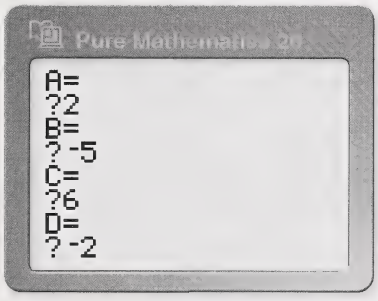
## Enrichment

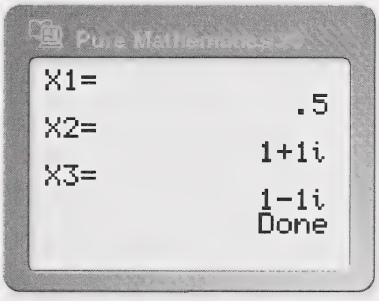
**1.**



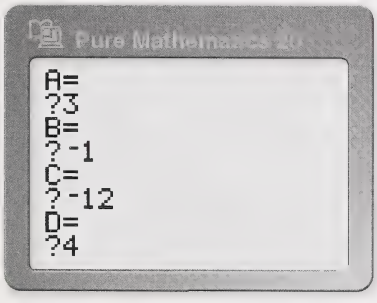



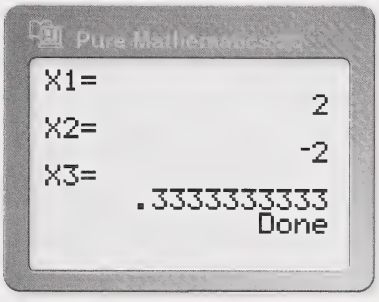

**2.**




**3.**



## Section 2: Activity 1

**1. a. Textbook question "Explore: Complete the Table", p. 242**

| | Operation | Resulting Function, $h(x)$ | $h(4)$ |
|---|---|---|---|
| A | $f(x)+g(x)$ | $h(x)=x^2+2x+1$ | 25 |
| B | $f(x)-g(x)$ | $h(x)=x^2-1$ | 15 |
| C | $f(x)\times g(x)$ | $h(x)=x^3+2x^2+x$ | 100 |
| D | $\frac{f(x)}{g(x)}$ | $h(x)=x;\ x\neq -1$ | 4 |

**b. Textbook questions 1 to 7 of "Inquire," p. 242**

**1.** $f(x)=x^2+x$ $\qquad$ $g(x)=x+1$

$f(4)=4^2+4$ $\qquad$ $g(4)=4+1$

$=16+4$ $\qquad$ $=5$

$=20$

**2. a.** $f(4)+g(4)=20+5$

$=25$

**b.** $f(4)-g(4)=20-5$

$=15$

**c.** $f(4)\times g(4)=20\times 5$

$=100$

**d.** $\frac{f(4)}{g(4)}=\frac{20}{5}$

$=4$

**3.** The values of $h(4)$ in A, B, C, and D are the same as the values found in question 2.

**4.** In D, the restriction on the function $g(x)$ is $x\neq -1$. This is because division by 0 in $\frac{f(x)}{g(x)}$ is undefined.

**5.** The ways in which functions are combined are the same as the ways in which real numbers are added, subtracted, multiplied, and divided.

**6. a.** $r(x)=p(x)+q(x)$
$=(2x^2+4x)+(2x)$
$=2x^2+6x$

**b.** $r(x)=p(x)-q(x)$
$=(2x^2+4x)-(2x)$
$=2x^2+2x$

**c.** $r(x)=q(x)-p(x)$
$=(2x)-(2x^2+4x)$
$=2x-2x^2-4x$
$=-2x^2-2x$

**d.** $r(x)=p(x)\times q(x)$
$=(2x^2+4x)(2x)$
$=4x^3+8x^2$

**e.** $r(x)=\dfrac{p(x)}{q(x)}$, where $q(x)\neq 0$
$=\dfrac{2x^2+4x}{2x}$
$=x+2$, where $x\neq 0$

**7. a.** $\ell(x)=\dfrac{A(x)}{w(x)}$, where $w(x)\neq 0$
$=\dfrac{11x^2-8x-3}{x-1}$
$=\dfrac{(11x+3)\cancel{(x-1)}}{\cancel{x-1}}$
$=11x+3$, where $x\neq 1$

**b.** $\ell(x)=11x+3$ $\quad$ $w(x)=x-1$
$\ell(7)=11(7)+3$ $\quad$ $w(7)=7-1$
$=80$ $\quad$ $=6$

The ballpark is 80 m × 6 m.

**2. Textbook questions 1, 7, 10, 27, 29, and 35 of "Practice," p. 247**

**1. a.** $(f+g)(x)=f(x)+g(x)$
$=(3x-4)+(x+6)$
$=4x+2$

**b.** $(f-g)(x)=f(x)-g(x)$
$=(3x-4)-(x+6)$
$=3x-4-x-6$
$=2x-10$

**c.** $(fg)(x)=f(x)\times g(x)$
$=(3x-4)(x+6)$
$=3x^2+18x-4x-24$
$=3x^2+14x-24$

**d.** $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$, where $g(x)\neq 0$
$=\dfrac{3x-4}{x+6}$, where $x\neq -6$

## Section 2: Activity 1 (continued)

**7. a.** $(f+g)(x)=3x+(-3x)$

$=0$

**b.** $(f-g)(x)=f(x)-g(x)$

$=3x-(-3x)$

$=3x+3x$

$=6x$

**c.** $(fg)(x)=f(x)\times g(x)$

$=(3x)(-3x)$

$=-9x^2$

**d.** $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$

$=\frac{3x}{-3x}$

$=-1$, where $x\neq 0$

**10. a.** $(f+g)(x)=f(x)+g(x)$

$=(2x+3)+(x-1)$

$=3x+2$

$(f+g)(3)=3(3)+2$

$=9+2$

$=11$

**b.** $(f-g)(x)=f(x)-g(x)$

$=(2x+3)-(x-1)$

$=2x+3-x+1$

$=x+4$

$(f-g)(-1)=-1+4$

$=3$

**c.** $(fg)(x)=f(x)\times g(x)$

$=(2x+3)(x-1)$

$=2x^2-2x+3x-3$

$=2x^2+x-3$

$(fg)(2)=2(2)^2+2-3$

$=8+2-3$

$=7$

**d.** $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$

$=\frac{2x+3}{x-1}$, where $x\neq 1$

$\left(\frac{f}{g}\right)(-2)=\frac{2(-2)+3}{-2-1}$

$=\frac{-4+3}{-3}$

$=\frac{-1}{-3}$

$=\frac{1}{3}$

**27. a.** $(f+g)(x)=f(x)+g(x)$

$=-2x^2+3x$

**b.** $(f-g)(x)=f(x)-g(x)$

$=-2x^2-3x$

**c.** $(fg)(x) = f(x) \times g(x)$
$= \left(-2x^2\right)(3x)$
$= -6x^3$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$
$= \frac{-2x^2}{3x}$
$= \frac{-2x}{3}$, where $x \neq 0$

**29. a.** $(f+g)(x) = f(x) + g(x)$
$= 12 + \left(x^2 - 3x + 2\right)$
$= x^2 - 3x + 14$

**b.** $(f-g)(x) = f(x) - g(x)$
$= 12 - \left(x^2 - 3x + 2\right)$
$= 12 - x^2 + 3x - 2$
$= -x^2 + 3x + 10$

**c.** $(fg)(x) = f(x) \times g(x)$
$= 12\left(x^2 - 3x + 2\right)$
$= 12x^2 - 36x + 24$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$
$= \frac{12}{x^2 - 3x + 2}$
$= \frac{12}{(x-1)(x-2)}$, where $x \neq 1, 2$

**35. a.** $(f+g)(x) = f(x) + g(x)$
$= \left(2x^2 - 4x\right) + (x - 2)$
$= 2x^2 - 3x - 2$
$(f+g)(-1) = 2(-1)^2 - 3(-1) - 2$
$= 2 + 3 - 2$
$= 3$

**b.** $(f-g)(x) = f(x) - g(x)$
$= \left(2x^2 - 4x\right) - (x - 2)$
$= 2x^2 - 4x - x + 2$
$= 2x^2 - 5x + 2$
$(f-g)(2) = 2(2)^2 - 5(2) + 2$
$= 8 - 10 + 2$
$= 0$

## Section 2: Activity 1 (continued)

**c.** $(fg)(x) = f(x) \times g(x)$

$= (2x^2 - 4x)(x - 2)$

$= 2x^3 - 4x^2 - 4x^2 + 8x$

$= 2x^3 - 8x^2 + 8x$

$(fg)(0) = 2(0)^3 - 8(0)^2 + 8(0)$

$= 0$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

$= \frac{2x^2 - 4x}{x - 2}$

$= \frac{2x(x-2)}{x-2}$

$= 2x$, where $x \neq 2$

$\left(\frac{f}{g}\right)(3) = 2(3)$

$= 6$

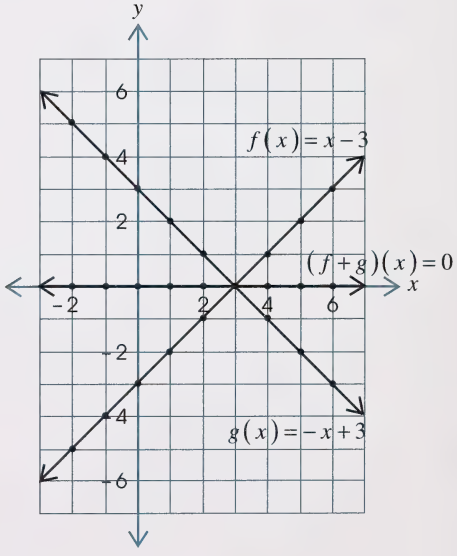
**3. Textbook questions 15 and 17 of "Practice," p. 247**

**15.** Given $f(x) = x - 3$ and $g(x) = -x + 3$, use their graphs to graph the sum $(f + g)(x)$.

Make a table of values for $f(x)$ and $g(x)$; then graph these functions on the same set of axes.

For each value of $x$, add the values of $f(x)$ and $g(x)$ to find the values of $(f + g)(x)$. Graph these ordered pairs for $(f + g)(x)$ and join them with a line.

| $x$ | $f(x)$ | $g(x)$ | $(f+g)(x)$ |
|---|---|---|---|
| −2 | −5 | 5 | 0 |
| −1 | −4 | 4 | 0 |
| 0 | −3 | 3 | 0 |
| 1 | −2 | 2 | 0 |
| 2 | −1 | 1 | 0 |
| 3 | 0 | 0 | 0 |
| 4 | 1 | −1 | 0 |
| 5 | 2 | −2 | 0 |
| 6 | 3 | −3 | 0 |



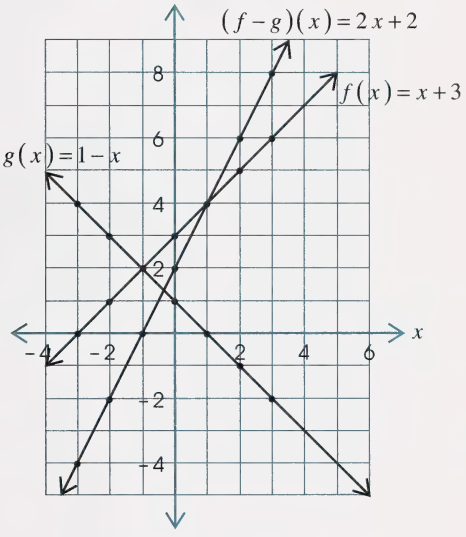

$\therefore (f + g)(x) = 0$

**17.** Given $f(x)=x+3$ and $g(x)=1-x$, use their graphs to graph the difference $(f-g)(x)$.

Make a table of values for $f(x)$ and $g(x)$; then graph these functions on the same set of axes.

For each value of $x$, subtract the values of $g(x)$ from the value of $f(x)$ to find the value of $(f-g)(x)$. Graph the ordered pairs for $(f-g)(x)$ and join them with a line.

| $x$ | $f(x)$ | $g(x)$ | $(f-g)(x)$ |
|---|---|---|---|
| $-3$ | 0 | 4 | $-4$ |
| $-2$ | 1 | 3 | $-2$ |
| $-1$ | 2 | 2 | 0 |
| 0 | 3 | 1 | 2 |
| 1 | 4 | 0 | 4 |
| 2 | 5 | $-1$ | 6 |
| 3 | 6 | $-2$ | 8 |




$$\begin{aligned}\therefore\ (f-g)(x) &= f(x)-g(x)\\ &= (x+3)-(1-x)\\ &= x+3-1+x\\ &= 2x+2\end{aligned}$$

**4.** When $F(h)=11°\text{C}$

$$\begin{aligned}F(h) &= 20-0.002h,\ \text{where } h\le 10\,000\\ 11 &= 20-0.002h\\ -9 &= -0.002h\\ h &= \frac{9}{0.002}\\ &= 4500\end{aligned}$$

The temperature is $11°\text{C}$ at an altitude of 4500 m.

## Section 2: Activity 1 (continued)

**5. a. Textbook questions 36, 39, 40, 43, 44.a., 46.a., and 53 of "Practice," p. 248**

**36. a.** $(f+g)x = f(x)+g(x)$

$$= \frac{1}{x+1}+\frac{1}{x}$$

$$= \frac{x}{x(x+1)}+\frac{x+1}{x(x+1)}$$

$$= \frac{x+(x+1)}{x(x+1)}$$

$$= \frac{2x+1}{x^2+x}, \text{ where } x \neq -1, 0$$

**b.** $(f-g)x = f(x)-g(x)$

$$= \frac{1}{x+1}-\frac{1}{x}$$

$$= \frac{x}{x(x+1)}-\frac{x+1}{x(x+1)}$$

$$= \frac{x-(x+1)}{x(x+1)}$$

$$= \frac{x-x-1}{x^2+x}$$

$$= \frac{-1}{x^2+x}, \text{ where } x \neq -1, 0$$

**c.** $(fg)(x) = f(x)\times g(x)$

$$= \frac{1}{x+1}\times\frac{1}{x}$$

$$= \frac{1}{x^2+x}, \text{ where } x \neq -1, 0$$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ where } g(x) \neq 0$

$$= \frac{\frac{1}{x+1}}{\frac{1}{x}}$$

$$= \frac{1}{x+1}\times\frac{x}{1}$$

$$= \frac{x}{x+1}, \text{ where } x \neq -1, 0$$

**39. a.** $(f+g)(x) = f(x)+g(x)$

$$= \frac{x}{x-4}+\frac{2x}{x+4}$$

$$= \frac{x(x+4)+2x(x-4)}{(x-4)(x+4)}$$

$$= \frac{x^2+4x+2x^2-8x}{x^2-16}$$

$$= \frac{3x^2-4x}{x^2-16}, \text{ where } x \neq \pm 4$$

**b.** $(f-g)(x) = f(x)-g(x)$

$$= \frac{x}{x-4}-\frac{2x}{x+4}$$

$$= \frac{x(x+4)-2x(x-4)}{(x-4)(x+4)}$$

$$= \frac{x^2+4x-2x^2+8x}{x^2-16}$$

$$= \frac{-x^2+12x}{x^2-16}, \text{ where } x \neq \pm 4$$

**c.** $(fg)(x) = f(x) \times g(x)$

$$= \frac{x}{x-4} \times \frac{2x}{x+4}$$

$$= \frac{2x^2}{x^2-16}, \text{ where } x \neq \pm 4$$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

$$= \frac{\frac{x}{x-4}}{\frac{2x}{x+4}}$$

$$= \frac{1}{x-4} \times \frac{x+4}{2}$$

$$= \frac{x+4}{2x-8}, \text{ where } x \neq 0, \pm 4$$

**40. a.** $(f+g)(x) = f(x) + g(x)$

$$= 3\sqrt{x} + \sqrt{x}$$

$$= 4\sqrt{x}, \text{ where } x \geq 0$$

*Note: Negative values are excluded because $f(x)$ and $g(x)$ are undefined when $x < 0$.*

**b.** $(f-g)(x) = f(x) - g(x)$

$$= 3\sqrt{x} - \sqrt{x}$$

$$= 2\sqrt{x}, \text{ where } x \geq 0$$

**c.** $(fg)(x) = f(x) \times g(x)$

$$= 3\sqrt{x} \times \sqrt{x}$$

$$= 3x, \text{ where } x \geq 0$$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

$$= \frac{3\sqrt{x}}{\sqrt{x}}$$

$$= 3, \text{ where } x > 0$$

**43. a.** $(f+g)(x) = f(x) + g(x)$

$$= \left(2\sqrt{x} - 1\right) + \left(\sqrt{x} + 1\right)$$

$$= 3\sqrt{x}, \text{ where } x \geq 0$$

**b.** $(f-g)(x) = f(x) - g(x)$

$$= \left(2\sqrt{x} - 1\right) - \left(\sqrt{x} + 1\right)$$

$$= 2\sqrt{x} - 1 - \sqrt{x} - 1$$

$$= \sqrt{x} - 2, \text{ where } x \geq 0$$

## Section 2: Activity 1 (continued)

**c.** $(fg)(x) = f(x) \times g(x)$

$= \left(2\sqrt{x} - 1\right)\left(\sqrt{x} + 1\right)$

$= 2x + 2\sqrt{x} - \sqrt{x} - 1$

$= 2x + \sqrt{x} - 1$, where $x \geq 0$

**d.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

$= \frac{2\sqrt{x} - 1}{\sqrt{x} + 1}$, where $x \geq 0$

**44. a.** $3f(x) + 2g(x) = 3(2x - 3) + 2(4 - x)$

$= 6x - 9 + 8 - 2x$

$= 4x - 1$

**46. a.** $2f(x) - 3g(x) = 2\left(2x^2\right) - 3(x + 3)$

$= 4x^2 - 3x - 9$

**53.** $f(x) - g(x) + \frac{1}{2}h(x) = \left(x^2 - 6x - 4\right) - \left(5x^2 - 9\right) + \frac{1}{2}(2x - 6)$

$= x^2 - 6x - 4 - 5x^2 + 9 + x - 3$

$= -4x^2 - 5x + 2$

**b. Textbook questions 54, 57, 61, 67.a., 69, and 75 of "Applications and Problem Solving," pp. 248 to 250**

**54. a.** Joseph's pay for the first 37.5 h is $37.5 \times \$12 = \$450$.

$\therefore r(x) = 450$

Joseph's pay for overtime hours is $15(x - 37.5)$, where $x \geq 37.5$.

$\therefore p(x) = 15(x - 37.5)$

$= 15x - 562.5$, where $x \geq 37.5$

**b.** Joseph's total pay $t(x)$ = Joseph's regular pay $r(x)$ + Joseph's overtime pay $p(x)$

$\therefore t(x) = r(x) + p(x)$

$= 450 + 15x - 562.5$

$= 15x - 112.5$, where $x \geq 37.5$

**c.** The domain of $t(x)$ is $x \geq 37.5$ because Joseph works at least 37.5 h every week.

**d.** $t(x) = 15x - 112.5$

$t(43) = 15(43) - 112.5$

$= 645 - 112.5$

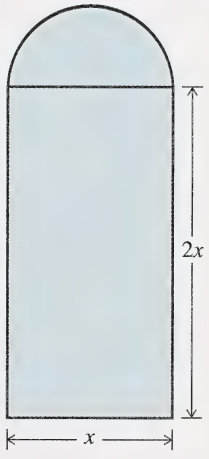
$= 532.5$

Joseph's total pay for the week is \$532.50.

**57. a.** $r(x) =$ area of rectangular part

$= 2x(x)$

$= 2x^2$




**b.** $c(x) =$ area of semi-circular part

$$= \frac{1}{2}\left(\pi r^2\right)$$

$$= \frac{1}{2}\left(\pi\left(\frac{x}{2}\right)^2\right)$$

$$= \frac{1}{2}\left(\frac{\pi x^2}{4}\right)$$

$$= \frac{\pi x^2}{8}$$

**c.** $w(x) =$ area of whole window

$= r(x) + c(x)$

$$= 2x^2 + \frac{\pi x^2}{8}$$

$$= \frac{16x^2 + \pi x^2}{8}$$

$$= \left(\frac{16+\pi}{8}\right)x^2$$

## Section 2: Activity 1 (continued)

**61.** **a.** $B(r) = \text{area of the base}$

$= \pi r^2$

**b.** $h(r) = \text{height}$

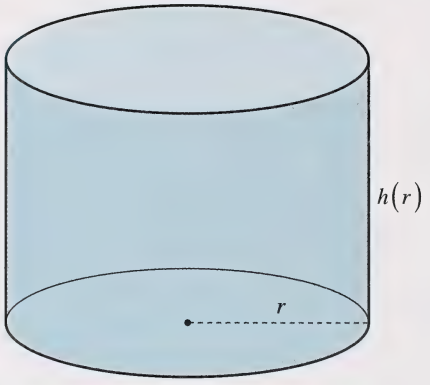
$= 3r$

**c.** $V(r) = \text{volume}$

$= B(r) \times h(r)$

$= \pi r^2 (3r)$

$= 3\pi r^3$




**d.** $V(r) = 3\pi r^3$

$V(7) = 3\pi (7)^3$

$\doteq 3233$

The volume of the cylinder is approximately 3233 $\text{cm}^3$.

**67.** **a.** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

$= \frac{6x^2}{2x}$

$= 3x$, where $x \neq 0$

But $h(x) = 3x$.

The domain of $\left(\frac{f}{g}\right)(x)$ is $x \neq 0$, whereas the domain of $h(x)$ is the set of reals. Because the domains differ, $\left(\frac{f}{g}\right)(x)$ is not the same function as $h(x)$.

**69.** **a.** $f(x) = 2x + 3$ and $g(x) = x^2 - 4$

**b.** The formula in cell D1 should be "= B1 + C1."

c.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | = +2*A1 + 3 | = +A1^2 – 4 | = B1 + C1 |
| 2 | 1 | 5 | –3 | 2 |
| 3 | –1 | 1 | –3 | –2 |
| 4 | 2 | 7 | 0 | 7 |
| 5 | –2 | –1 | 0 | –1 |

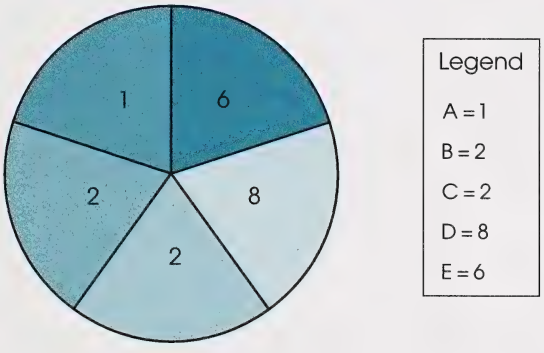
**75.** Because $(f+g)(x)=f(x)+g(x)$,

$$g(x)=(f+g)(x)-f(x)$$

If $(f+g)(x)$ and $f(x)$ were graphed on the same set of axes, subtract the $y$-value of $f(x)$ from the $y$-value of $(f+g)(x)$ for different values of $x$ to find $g(x)$. Plot the resulting ordered pairs for $g(x)$, and join them with a smooth curve.

**6. Textbook question "NUMBER POWER," p. 250**

Answers may vary. A sample answer is given.




1 = A
2 = B
3 = A + B
4 = B + C
5 = A + B + C
6 = E
7 = A + E
8 = D
9 = E + A + B
10 = C + D
11 = E + A + B + C
12 = B + C + D
13 = A + B + C + D
14 = D + E
15 = D + E + A
16 = C + D + E
17 = C + D + E + A
18 = B + C + D + E
19 = A + B + C + D + E

## Section 2: Activity 2

**1. a. Textbook questions a. to c. of "Explore: Use the Formulas," p. 251**

**a.**

$$g(d) = 220d$$
$$g(1000) = 220(1000)$$
$$= 220\,000$$

Rohana received 220 000 drachmas.

**b.**

$$\ell(g) = 6g$$
$$\ell(120\,000) = 6(120\,000)$$
$$= 720\,000$$

Rohana received 720 000 lira.

**c.**

$$\ell(g(d)) = \ell(220d)$$
$$\ell(g(200)) = \ell(220(200))$$
$$= \ell(44\,000)$$
$$= 6(44\,000)$$
$$= 264\,000$$

Rohana received 264 000 lira.

**b. Textbook questions 1 to 5 of "Inquire," p. 251**

**1.**

$$\ell(g) = 6g$$
$$\ell(d) = 6(220d) \quad \leftarrow g(d) = 220d$$
$$= 1320d$$

**2.**

$$\ell = 6g$$
$$g = \frac{1}{6}\ell$$
$$\therefore g(\ell) = \frac{1}{6}\ell$$

**3.**

$$g = 220d$$
$$d = \frac{1}{220}g$$
$$\therefore d(g) = \frac{1}{220}g$$

**4.**

$$d = \frac{1}{220}g$$
$$d = \frac{1}{220}\left(\frac{1}{6}\ell\right) \quad \leftarrow g(\ell) = \frac{1}{6}\ell$$
$$= \frac{1}{1320}\ell$$
$$\therefore d(\ell) = \frac{1}{1320}\ell$$

**5. a.** $j(f) = 23f$

$j(d) = 23(4d)$ ← $f(d) = 4d$

$= 92d$

**b.** $j = 92d$

$d = \frac{1}{92}j$

$\therefore\ d(j) = \frac{1}{92}j$

**2. a. Textbook questions 1 to 10 of "Practice," p. 256**

**1.** $f(5) = 12$

$\therefore\ g(f(5)) = g(12)$

$= 4$

**2.** $f(3) = 2$

$\therefore\ g(f(3)) = g(2)$

$= 2$

**3.** $f(1) = 0$

$\therefore\ g(f(1)) = g(0)$

$= 2$

**4.** $f(4) = 7$

$\therefore\ g(f(4)) = g(7)$

$= 5$

**5.** $g(12) = 4$

$\therefore\ f(g(12)) = f(4)$

$= 7$

**6.** $g(2) = 2$

$\therefore\ f(g(2)) = f(2)$

$= -3$

**7.** $g(7) = 5$

$\therefore\ f(g(7)) = f(5)$

$= 12$

**8.** $g(-3) = 1$

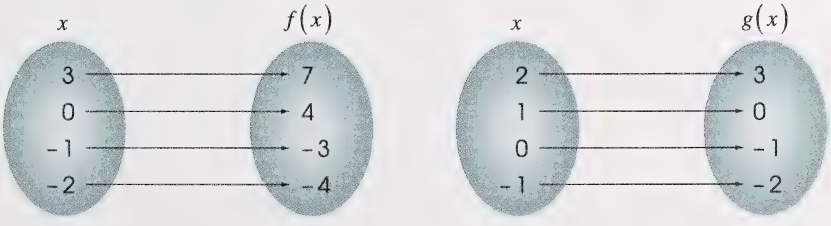
$\therefore\ f(g(-3)) = f(1)$

$= 0$

**9. a.** $g \circ f = \{(1,2), (2,1), (3,2), (4,5), (5,4)\}$

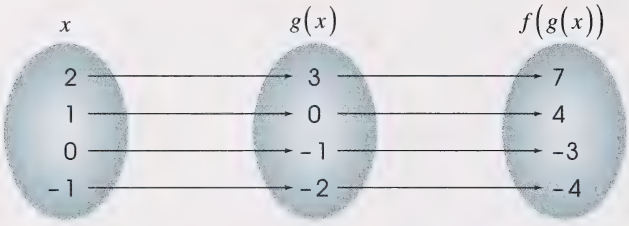
**b.** $f \circ g = \{(-3,0), (0,-3), (2,-3), (7,12), (12,7)\}$

**10.**



## Section 2: Activity 2 (continued)

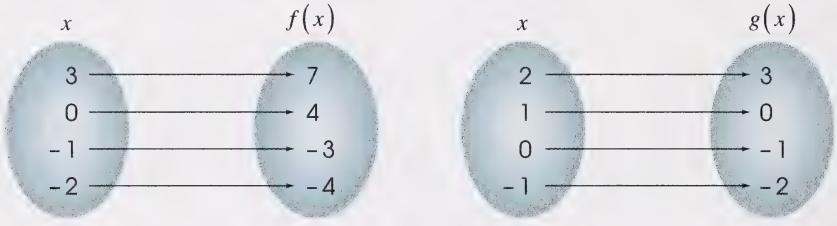
**a.** For $f \circ g$, map $g$ first and then $f$.




The outputs from $g$ are acceptable inputs for $f$.

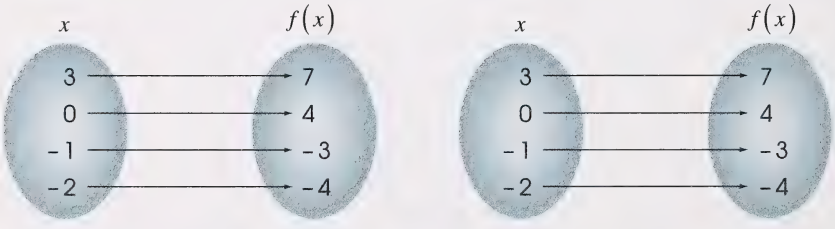
$\therefore f \circ g = \{(2,7),\ (1,4),\ (0,-3),\ (-1,-4)\}$

**b.** For $g \circ f$, map $f$ first and then $g$.



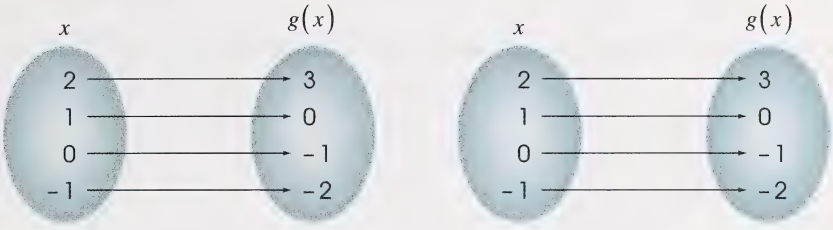

The outputs from $f$ are not acceptable inputs for $g$. Therefore, $g \circ f$ does not exist.

**c.** For $f \circ f$, map $f$ twice.




The outputs of $f$ are not acceptable inputs for $f$. Therefore, $f \circ f$ does not exist.

**d.** For $g \circ g$, map $g$ twice.




The outputs of $g$ are not acceptable inputs for $g$. Therefore, $g \circ g$ does not exist.

**b. Textbook questions 12, 14, 16, and 18 of "Practice," p. 256**

**12.**
$$g(x) = 2x$$
$$g(3) = 2(3)$$
$$= 6$$

$$\therefore f(g(3)) = f(6)$$
$$= 6+3$$
$$= 9$$

**14.**
$$g(x) = 2x$$
$$g(4) = 2(4)$$
$$= 8$$

$$\therefore (f \circ g)(4) = f(g(4))$$
$$= f(8)$$
$$= 8+3$$
$$= 11$$

**16.**
$$g(x) = 4x$$
$$g(-2) = 4(-2)$$
$$= -8$$

$$\therefore (f \circ g)(-2) = f(g(-2))$$
$$= f(-8)$$
$$= 2(-8)-3$$
$$= -16-3$$
$$= -19$$

**18.**
$$g(x) = 1-x$$
$$g(3) = 1-3$$
$$= -2$$

$$\therefore (f \circ g)(3) = f(g(3))$$
$$= f(-2)$$
$$= 4-5(-2)$$
$$= 4+10$$
$$= 14$$

## Section 2: Activity 2 (continued)

**3. a. Textbook questions 22, 31, 39, 50, 51, 61, 64, and 68 of "Practice," pp. 256 and 257**

**22.** $(f \circ g)(x) = f(g(x))$
$= f(100 - x)$
$= 5(100 - x)$
$= 500 - 5x$

$(g \circ f)(x) = g(f(x))$
$= g(5x)$
$= 100 - 5x$

**31.** $(f \circ g)(x) = f(g(x))$
$= f(2x - 1)$
$= 4 - (2x - 1)^2$
$= 4 - (4x^2 - 4x + 1)$
$= 4 - 4x^2 + 4x - 1$
$= -4x^2 + 4x + 3$

$(g \circ f)(x) = g(f(x))$
$= g(4 - x^2)$
$= 2(4 - x^2) - 1$
$= 8 - 2x^2 - 1$
$= -2x^2 + 7$

**39. a.** $f(x) = 2 - x$
$f(3) = 2 - 3$
$= -1$

$\therefore (f \circ f)(3) = f(f(3))$
$= f(-1)$
$= 2 - (-1)$
$= 3$

**b.** $(f \circ f)(x) = f(f(x))$
$= f(2 - x)$
$= 2 - (2 - x)$
$= 2 - 2 + x$
$= x$

**50.** $(f \circ g)(x) = f(g(x))$
$= f(|2x|)$
$= \sqrt{-|2x|}$

This composite function does not exist because the square root of a negative number is non-real.

**51.** $(h \circ g)(x) = h(g(x))$
$= h(\sqrt{x})$
$= 3\sqrt{x} + 2$

The range of $h \circ g$ is $y \geq 2$. Because the least value of $\sqrt{x}$ is 0, the least value of $y$ is $3\sqrt{0} + 2 = 2$.

**61.** $(f \circ g)(x) = f(g(x))$
$= f(2x)$
$= \frac{2x}{2x - 2}$
$= \frac{2x}{2(x-1)}$
$= \frac{x}{x-1}$, where $x \neq 1$

$(g \circ f)(x) = g(f(x))$
$= g\left(\frac{x}{x-2}\right)$
$= 2\left(\frac{x}{x-2}\right)$
$= \frac{2x}{x-2}$, where $x \neq 2$

**64. a.** $(h \circ k)(x) = h(k(x))$
$= h(4x)$
$= \sqrt{4x+1}$

$(k \circ h)(x) = k(h(x))$
$= k\left(\sqrt{x+1}\right)$
$= 4\sqrt{x+1}$

**b.** The domain of $h$ is $x \geq -1$.

The domain of $k$ is the set of reals.

The domain of $h \circ k$ is

$4x + 1 \geq 0$
$4x \geq -1$
$x \geq -\frac{1}{4}$

The domain of $k \circ h$ is $x \geq -1$.

**c.** The range of $h \circ k$ is $y \geq 0$.

The range of $k \circ h$ is $y \geq 0$.

## Section 2: Activity 2 (continued)

**68. a.** $(f \circ g)(x) = f(g(x))$
$= f(|x|)$
$= 2|x|$

$(g \circ f)(x) = g(f(x))$
$= g(2x)$
$= |2x|$
$= 2|x|$

**b.** The domains of $f$, $g$, $f \circ g$, and $g \circ f$ are the set of real numbers.

**c.** The ranges of $f \circ g$ and $g \circ f$ are $y \geq 0$.

**b. Textbook questions 69, 72, 73, 88, 91, and 92 of "Applications and Problem Solving," pp. 257 to 259**

**69. a.** $s(p) = 16p$
$s(p(d)) = 16(0.5d)$
$= 8d$

**b.** $s(p(1000)) = 8(1000)$
$= 8000$

Therefore, \$1000 Canadian is 8000 Austrian schillings.

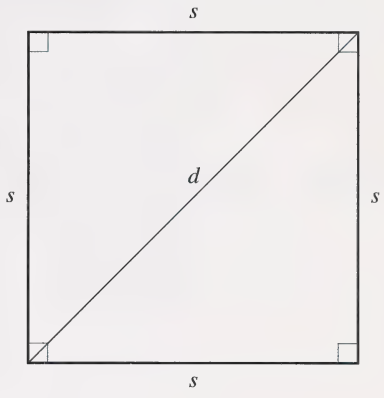
**c.** $s = 8d$
$d = \frac{1}{8}s$

Therefore, $d(s) = \frac{1}{8}s$ converts Austrian schillings directly to Canadian dollars.

**d.** $d(1000) = \frac{1}{8}(1000)$
$= 125$

Therefore, 1000 schillings is \$125.

**72. a.** $A = s^2$ or $A(s) = s^2$




**b.** Use the Pythagorean Theorem to find the relationship between the length of the diagonal, $d$, and the length of a side, $s$.

$$d^2 = s^2 + s^2$$
$$d^2 = 2s^2$$
$$d = \sqrt{2}s$$
$$s = \frac{1}{\sqrt{2}}d$$

$$\therefore\ s(d) = \frac{1}{\sqrt{2}}d \text{ or } \frac{\sqrt{2}}{2}d$$

**c.** $A(s) = s^2$

$$A(s(d)) = \left(\frac{1}{\sqrt{2}}d\right)^2$$
$$= \frac{d^2}{2}$$

$$\left(\frac{\sqrt{2}}{2}d\right)^2 = \frac{2d^2}{4}$$
$$= \frac{d^2}{2}$$

**d.** $A(s(12)) = \frac{12^2}{2}$

$$= \frac{144}{2}$$
$$= 72$$

The area of the square is $72\ \text{cm}^2$.

**73. a.** distance = speed × time

$$d(t) = 330t$$

**b.** $A(d) = \pi d^2$

*Note: In question 73.b., the radius is represented by d, not r.*

## Section 2: Activity 2 (continued)

**c.** $A(d) = \pi d^2$

$A(d(t)) = \pi(330t)^2$

$A(t) = 108\,900\pi t^2$

**d.** $A(0.1) = 108\,900\pi(0.1)^2$

$= 1089\pi$

$\doteq 3420$

The area is approximately 3420 $\text{m}^2$.

**e.** Convert 1 $\text{km}^2$ to $\text{m}^2$.

$1\ \text{km}^2 = 1(1000\ \text{m})^2$

$= 1\,000\,000\ \text{m}^2$

$\therefore A(t) = 108\,900\pi t^2$

$1\,000\,000 = 108\,900\pi t^2$

$t = \sqrt{\dfrac{1\,000\,000}{108\,900\pi}}$

$\doteq 1.7$

It takes sound approximately 1.7 s to be heard over an area of 1 $\text{km}^2$.

**88. a.** $j(r) = r - 20\%\ \text{of}\ r$

$= r - 0.20r$

$= 0.80r$

**b.** $s(r) = r - 40\%\ \text{of}\ r$

$= r - 0.40r$

$= 0.60r$

**c.** $j(s(r)) = j(0.60r)$
$= 0.80(0.60r)$
$= 0.48r$

$s(j(r)) = s(0.80r)$
$= 0.60(0.80r)$
$= 0.48r$

In the composite function $j(s(r))$, the sale price is determined first; then Jacob's employee discount is deducted from the sale price. In the composite function $s(j(r))$, Jacob's employee discount is calculated first, then 40% is deducted from the discounted price.

Either way, Jacob would pay 48% of the regular price.

**d.** $s(j(r)) = 0.48(40)$
$= 19.2$

Because $s(j(r)) = j(s(r))$, the order does not matter.

The hammer would cost Jacob \$19.20.

**e.** $m(r) = r - 5$

$j(r) = 0.80r$
$j(m(r)) = 0.80(r-5)$
$= 0.80r - 4$

$m(r) = r - 5$
$m(j(r)) = 0.80r - 5$

Because $j(m(r)) \neq m(j(r))$, the order of the composition matters.

If Jacob receives the \$5 manufacturer's discount first, his employee discount would be 20% of $(\$50 - \$5)$, or \$9. Therefore, he would pay $\$45 - \$9 = \$36$ for the kettle.

If Jacob receives his employee discount first, the discounted price would be \$50 – 20% of \$50, or \$40. If Jacob now receives the manufacture's discount of \$5, he would pay only \$35 for the kettle.

## Section 2: Activity 2 (continued)

**91.**
$$f(g(x)) = h(x)$$
$$2(g(x)) + 1 = 3 - 2x$$
$$2(g(x)) = 2 - 2x$$
$$g(x) = \frac{2 - 2x}{2}$$
$$= \frac{2(1-x)}{2}$$
$$= 1 - x$$

**Check**

| LS | RS |
|---|---|
| $f(g(x))$ | $h(x)$ |
| $= f(1-x)$ | $= 3-2x$ |
| $= 2(1-x)+1$ | |
| $= 2-2x+1$ | |
| $= 3-2x$ | |

LS = RS

**92. a.** This statement is always true. Suppose $f(x) = mx + b$ and $g(x) = nx + c$ are two linear functions.

$$\therefore f(g(x)) = f(nx + c)$$
$$= m(nx + c) + b$$
$$= mnx + mc + b$$
$$= mnx + (mc + b)$$
$$= px + q, \text{ where } p = mn \text{ and } q = mc + b$$

The composition function has the form of a linear function.

**b.** The product of two linear functions is never a linear function.

Suppose $f(x) = mx + b$ and $g(x) = nx + c$ are linear functions.

$$\therefore (fg)x = f(x) \times g(x)$$
$$= (mx + b)(nx + c)$$
$$= mnx^2 + bnx + cmx + bc$$

Because neither $m$ nor $n$ is 0, the composition is a quadratic function.

**c.** The composition of two quadratic functions is never a quadratic function.

Suppose $f(x)=a_1x^2+b_1x+c_1$ and $g(x)=a_2x^2+b_2x+c_2$ are quadratic functions. Therefore, $a_1\neq 0$ and $a_2\neq 0$.

$$\begin{aligned}\therefore f(g(x)) &= f\left(a_2x^2+b_2x+c_2\right)\\ &= a_1\left(a_2x^2+b_2x+c_2\right)^2+b_1\left(a_2x^2+b_2x+c_2\right)+c_1\\ &= a_1a_2x^4+\cdots+c_1\end{aligned}$$

Because neither $a_1$ nor $a_2$ is 0, the composition is a quartic function.

**d.** It is sometimes true that a function equals its composition with itself.

For example, if $f(x)=x$,

$$\begin{aligned}(f\circ f)(x) &= f(f(x))\\ &= f(x)\end{aligned}$$

However, if $f(x)=2x$,

$$\begin{aligned}(f\circ f)(x) &= f(f(x))\\ &= f(2x)\\ &= 2(2x)\\ &= 4x\\ &\neq f(x)\end{aligned}$$

**4. Textbook questions 1 to 12 of "Mental Math: Dividing Using Compatible Numbers," p. 241**

**1.** $$\begin{aligned}96\div 4 &= (100-4)\div 4\\ &= (100\div 4)-(4\div 4)\\ &= 25-1\\ &= 24\end{aligned}$$

**2.** $$\begin{aligned}132\div 12 &= (120+12)\div 12\\ &= (120\div 12)+(12\div 12)\\ &= 10+1\\ &= 11\end{aligned}$$

**3.** $$\begin{aligned}135\div 15 &= (150-15)\div 15\\ &= (150\div 15)-(15\div 15)\\ &= 10-1\\ &= 9\end{aligned}$$

**4.** $$\begin{aligned}143\div 11 &= (110+33)\div 11\\ &= (110\div 11)+(33\div 11)\\ &= 10+3\\ &= 13\end{aligned}$$

## Section 2: Activity 2 (continued)

**5.** $198 \div 22 = (220 - 22) \div 22$
$= (220 \div 22) - (22 \div 22)$
$= 10 - 1$
$= 9$

**6.** $228 \div 19 = (190 + 38) \div 19$
$= (190 \div 19) + (38 \div 19)$
$= 10 + 2$
$= 12$

**7.** $156 \div 13 = (130 + 26) \div 13$
$= (130 \div 13) + (26 \div 13)$
$= 10 + 2$
$= 12$

**8.** $406 \div 14 = (420 - 14) \div 14$
$= (420 \div 14) - (14 \div 14)$
$= 30 - 1$
$= 29$

**9.** $306 \div 17 = (340 - 34) \div 17$
$= (340 \div 17) - (34 \div 17)$
$= 20 - 2$
$= 18$

**10.** $550 \div 25 = (500 + 50) \div 25$
$= (500 \div 25) + (50 \div 25)$
$= 20 + 2$
$= 22$

**11.** $437 \div 23 = (460 - 23) \div 23$
$= (460 \div 23) - (23 \div 23)$
$= 20 - 1$
$= 19$

**12.** $312 \div 26 = (260 + 52) \div 26$
$= (260 \div 26) + (52 \div 26)$
$= 10 + 2$
$= 12$

## Section 2: Activity 3

**1. a. Textbook question "Explore: Complete the Table," p. 261**

| Distance, *d* (km) | Number of Laps, *n* |
|---|---|
| 5 | 12.5 |
| 3 | 7.5 |
| 1.5 | 3.75 |
| 0.8 | 2 |
| 0.4 | 1 |
| 0.2 | 0.5 |

**b. Textbook questions 1 to 5 of "Inquire," p. 261**

**1. a.** $d(n)=0.4n$

1 lap = 400 m = 0.4 km

**b.** The function $d(n)$ multiplies each input value by 0.4.

**c.** Using the preceding table, the ordered pairs are $(12.5,5)$, $(7.5,3)$, $(3.75,1.5)$, $(2,0.8)$, $(1,0.4)$, and $(0.5,0.2)$.

**2. a.** $d=0.4n$

$$n=\frac{d}{0.4}$$

$$\therefore\ n(d)=\frac{d}{0.4}$$

**b.** The function $n(d)$ divides each input value by 0.4.

**c.** Using the preceding table, the ordered pairs are $(5,12.5)$, $(3,7.5)$, $(1.5,3.75)$, $(0.8,2)$, $(0.4,1)$, and $(0.2,0.5)$.

**3.** The ordered pairs of $n(d)$ can be obtained by switching the coordinates of the ordered pairs of $d(n)$.

**4.** The functions $d(n)$ and $n(d)$ are inverse functions because the ordered pairs of one function can be obtained by switching the ordered pairs of the other function.

**5.** The domain of one function is the range of the other function.

**2.** Let $(x_1, y_1)=P(a,b)$ and $(x_2, y_2)=Q(b,a)$.

$$\therefore \text{ Midpoint of } PQ=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$=\left(\frac{a+b}{2},\frac{b+a}{2}\right)$$

$$=\left(\frac{a+b}{2},\frac{a+b}{2}\right)$$

## Section 2: Activity 3 (continued)

Because the coordinates of the midpoint are equal, the midpoint lies on the line $y = x$; therefore, $y = x$ bisects $PQ$.

Now, calculate the slope of $PQ$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{a-b}{b-a} \\ &= \frac{-1(-a+b)}{b-a} \\ &= \frac{-1(b-a)}{b-a} \\ &= -1 \end{aligned}$$

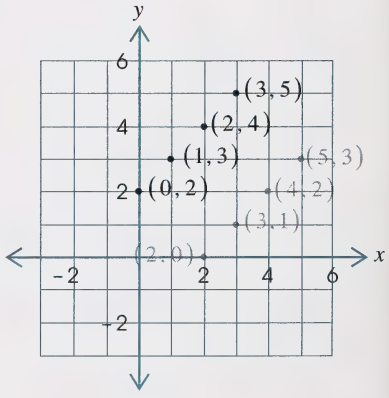
Because the slope of $PQ$ is $-1$ and the slope of $y = x$ is 1, the line $y = x$ intersects $PQ$ at a right angle. Therefore, $y = x$ is the perpendicular bisector of $PQ$.

**3. Textbook questions 1 to 4 of "Practice," p. 268**

**1.** Switch the first and second coordinates of each ordered pair.

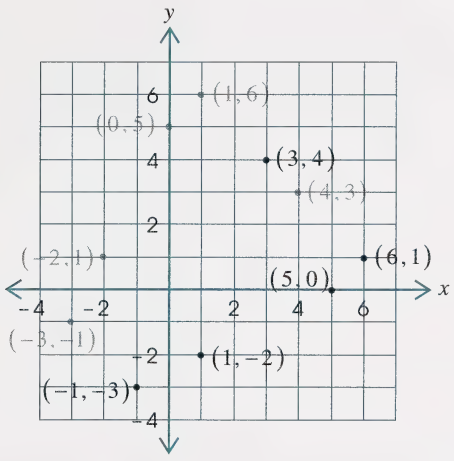
$$f^{-1} = \{(2,0),\ (3,1),\ (4,2),\ (5,3)\}$$

*Recall that the graphs of inverses will be grey throughout this module.*



**2.** Switch the first and second coordinates of each ordered pair.

$$g^{-1} = \{(-3,-1),\ (-2,1),\ (4,3),\ (0,5),\ (1,6)\}$$




**3.** Switch the first and second coordinates of each ordered pair.

$$f^{-1} = \{(3,-2),\ (2,-1),\ (0,0),\ (-2,4)\}$$

The inverse is a function because no two ordered pairs have the same $x$-coordinate.

**4.** Switch the first and second coordinates of each ordered pair.

$$g^{-1} = \{(-2,4),\ (1,2),\ (3,1),\ (-2,0),\ (-3,-3)\}$$

The inverse is not a function because the ordered pairs $(-2,\ 4)$ and $(-2,\ 0)$ have the same $x$-coordinate.

## Section 2: Activity 3 (continued)

**4. Textbook questions 11, 21, 26, 30, 31, 37, 45, and 47 of "Practice," p. 268**

**11.** $f(x) = x - 1$

$y = x - 1$ ← Replace $f(x)$ with $y$.

Interchange $x$ and $y$ to find the inverse.

$$x = y - 1$$
$$x + 1 = y$$
$$y = x + 1$$

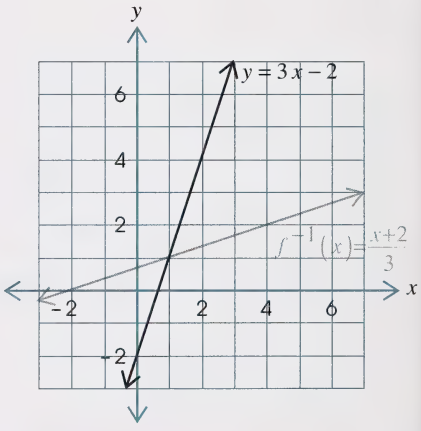
$\therefore f^{-1}(x) = x + 1$

**21.** $y = 3x - 2$

$x = 3y - 2$ ← Interchange $x$ and $y$ to find the inverse.

$$x + 2 = 3y$$
$$\frac{x+2}{3} = y$$
$$y = \frac{x+2}{3}$$

$\therefore f^{-1}(x) = \frac{x+2}{3}$




**26.** $f(x) = \frac{x+3}{4}$

$y = \frac{x+3}{4}$ ← Replace $f(x)$ with $y$.

$x = \frac{y+3}{4}$ ← Interchange $x$ and $y$ to find the inverse.

$$4x = y + 3$$
$$4x - 3 = y$$
$$y = 4x - 3$$

$\therefore f^{-1}(x) = 4x - 3$

The inverse is a linear function.

**30.** To determine if $f$ and $g$ are inverses, form the compositions of $f$ and $g$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then $f$ and $g$ are inverses.

**Check**

For $f \circ g$,

| LS | RS |
|---|---|
| $(f \circ g)(x)$ | $x$ |
| $= f(g(x))$ | |
| $= f\left(\frac{x}{7}\right)$ | |
| $= 7\left(\frac{x}{7}\right)$ | |
| $= x$ | |

LS = RS

For $g \circ f$,

| LS | RS |
|---|---|
| $(g \circ f)(x)$ | $x$ |
| $= g(f(x))$ | |
| $= g(7x)$ | |
| $= \frac{7x}{7}$ | |
| $= x$ | |

LS = RS

Therefore, $f$ and $g$ are inverses.

**31.** Form the compositions of $f$ and $g$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then $f$ and $g$ are inverses.

**Check**

For $f \circ g$,

| LS | RS |
|---|---|
| $(f \circ g)(x)$ | $x$ |
| $= f(g(x))$ | |
| $= f\left(\frac{x+1}{2}\right)$ | |
| $= 2\left(\frac{x+1}{2}\right) - 1$ | |
| $= x + 1 - 1$ | |
| $= x$ | |

LS = RS

For $g \circ f$,

| LS | RS |
|---|---|
| $(g \circ f)(x)$ | $x$ |
| $= g(f(x))$ | |
| $= g(2x-1)$ | |
| $= \frac{(2x-1)+1}{2}$ | |
| $= \frac{2x}{2}$ | |
| $= x$ | |

LS = RS

Therefore, $f$ and $g$ are inverses.

## Section 2: Activity 3 (continued)

**37. a.** $f(x) = x^2 + 1$

$y = x^2 + 1$ ← Replace $f(x)$ with $y$.

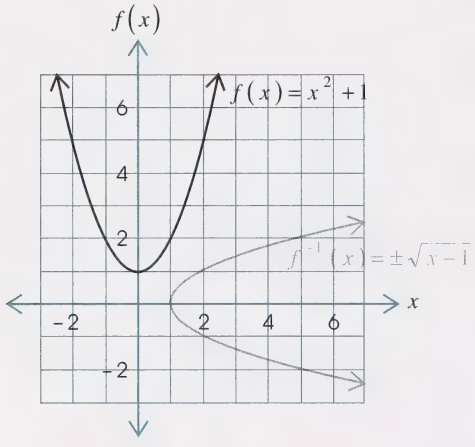
$x = y^2 + 1$ ← Interchange $x$ and $y$ to find the inverse.

$y^2 = x - 1$

$y = \pm\sqrt{x-1}$

$\therefore f^{-1}(x) = \pm\sqrt{x-1}$

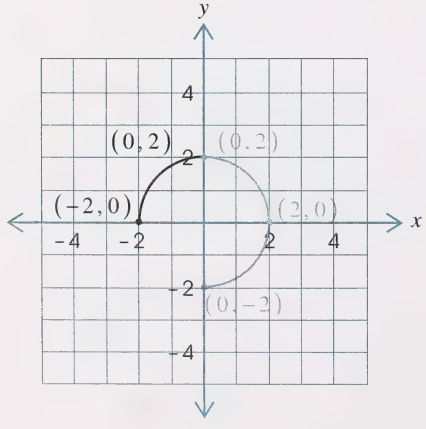
**b.**




**c.** The domain of $f(x)$ is the set of reals, and the range is $f(x) \geq 1$.

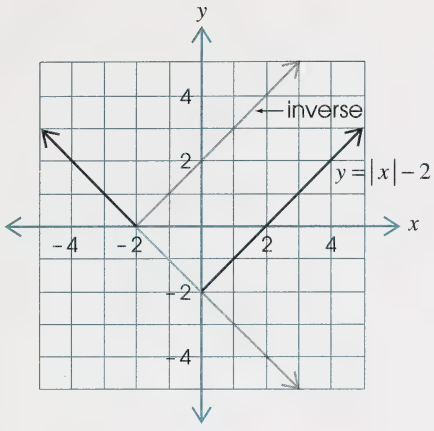
The domain of $f^{-1}(x)$ is $x \geq 1$, and the range is the set of reals.

**45.**




*Note: The colour portion of the graph shows where the graphs of $f(x)$ and its inverse overlap.*

**47.**




*Note:* The colour portion of the graph shows where the graphs of $f(x)$ and its inverse overlap.

**5. a. Textbook questions 68, 69, 75, 80, and 81 of "Practice," p. 269**

**68.**

$$y = \frac{1}{3+x}$$

$$x = \frac{1}{3+y}$$ ← Interchange $x$ and $y$ to find the inverse.

$$x(3+y) = 1$$

$$3x + xy = 1$$

$$xy = 1 - 3x$$

$$y = \frac{1-3x}{x}$$

$$= \frac{1}{x} - 3$$

$$\therefore f^{-1}(x) = \frac{1}{x} - 3$$

## Section 2: Activity 3 (continued)

**69.**

$$y = \frac{1-x}{x+2}$$

$$x = \frac{1-y}{y+2}$$ ← Interchange $x$ and $y$ to find the inverse.

$$x(y+2) = 1-y$$

$$xy + 2x = 1-y$$

$$xy + y = 1-2x$$

$$y(x+1) = 1-2x$$

$$y = \frac{1-2x}{x+1}$$

$$\therefore f^{-1}(x) = \frac{1-2x}{x+1}$$

**75.**

$$y = \sqrt{x^2+9}$$

$$x = \sqrt{y^2+9}$$ ← Interchange $x$ and $y$ to find the inverse.

$$x^2 = y^2 + 9, \text{ where } x \geq 3$$

$$x^2 - 9 = y^2$$

$$y = \pm\sqrt{x^2-9}$$

$$\therefore f^{-1}(x) = \pm\sqrt{x^2-9}, \text{ where } x \geq 3$$

*The range of the original function becomes the domain of the inverse.*

**80. a.**

$$f(x) = 1-x^2$$

$$y = 1-x^2$$ ← Replace $f(x)$ with $y$.

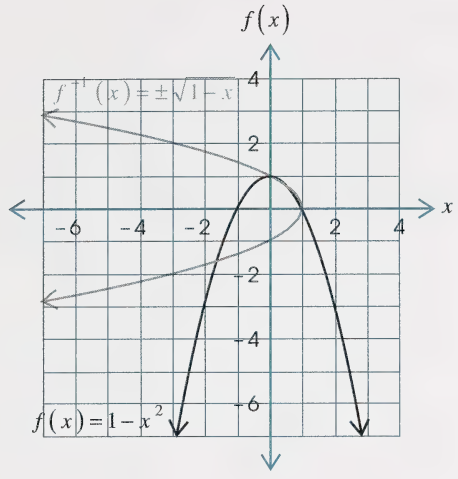
$$x = 1-y^2$$ ← Interchange $x$ and $y$ to find the inverse.

$$y^2 = 1-x$$

$$y = \pm\sqrt{1-x}$$

$$\therefore f^{-1}(x) = \pm\sqrt{1-x}$$

**b.**



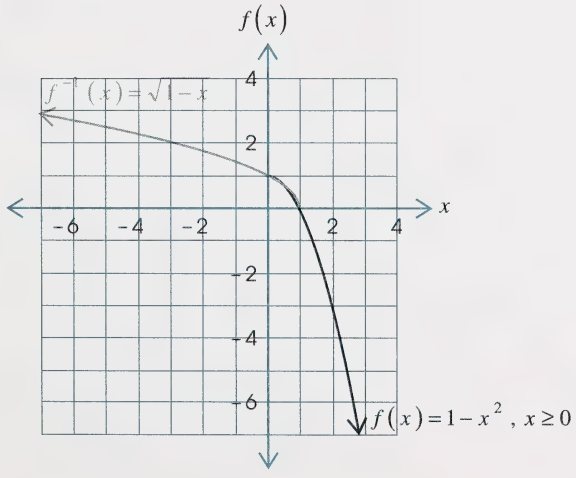

**c.** Answers may vary. A sample answer is given.

Restrict $f(x)$ so that $x \geq 0$. This will remove the left-hand half of the graph of $f(x)$ and the bottom half of $f^{-1}(x)$.

$f(x) = 1 - x^2$, where $x \geq 0$

$\therefore f^{-1}(x) = \sqrt{1-x}$

**d.**




**e.** The domain of $f$ is the set of all real numbers; the range is $y \leq 1$. The domain of $f^{-1}$ is $x \leq 1$; the range is the set of all real numbers.

## Section 2: Activity 3 (continued)

**81. a.** $f(x)=(x-2)^2$

$y=(x-2)^2$ ← Replace $f(x)$ with $y$.

$x=(y-2)^2$ ← Interchange $x$ and $y$ to find the inverse.

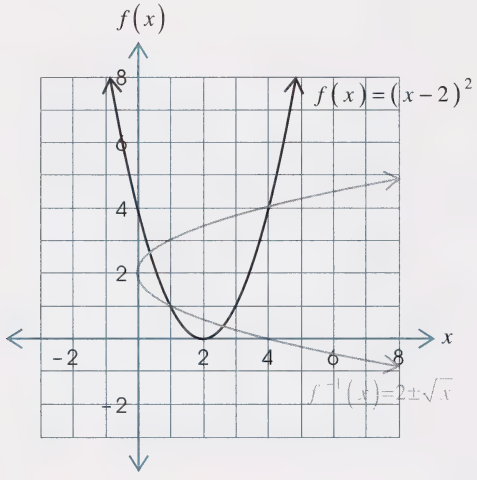
$\pm\sqrt{x}=y-2$

$2\pm\sqrt{x}=y$

$y=2\pm\sqrt{x}$

$\therefore f^{-1}(x)=2\pm\sqrt{x}$

**b.**



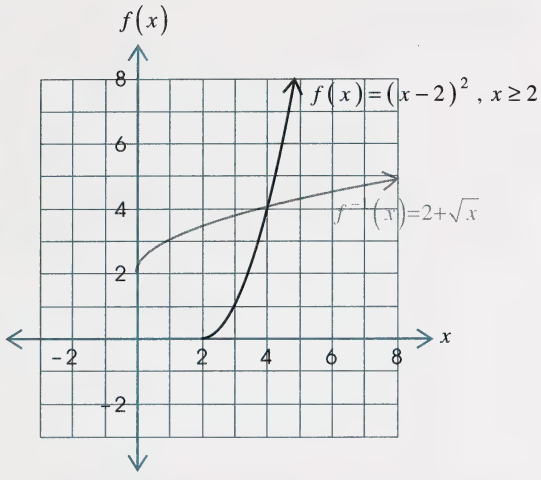

**c.** Answers may vary. A sample answer is given.

Restrict $f(x)$ so that $x\geq 2$. This will remove the left-hand half of the graph of $f(x)$ and the bottom half of the graph of $f^{-1}(x)$.

$f(x)=(x-2)^2$, where $x\geq 2$

$\therefore f^{-1}(x)=2+\sqrt{x}$

**d.**




**e.** The domain of $f$ is the set of all real numbers; the range is $y \geq 0$. The domain of $f^{-1}$ is $x \geq 0$; the range is the set of all real numbers.

**b. Textbook questions 84, 86, 89, 94, and 96 of "Applications and Problem Solving," pp. 269 and 270**

**84. a.** $f(x) = 2\pi x$, where $x \geq 0$

The variable $x$ is non-negative because it represents the radius of a circle.

**b.** $f(x) = 2\pi x$

$y = 2\pi x$ ← Replace $f(x)$ with $y$.

$x = 2\pi y$ ← Interchange $x$ and $y$ to find the inverse.

$$\frac{x}{2\pi} = y$$

$$y = \frac{x}{2\pi}$$

$$\therefore f^{-1}(x) = \frac{x}{2\pi}, \text{ where } x \geq 0$$

The variable $x$ is non-negative because it is the circumference of a circle.

**c.** The inverse is a function. For a given value of $x$ in the domain, there is exactly one $y$-value in the range.

**d.** The inverse represents finding the radius from the circumference.

## Section 2: Activity 3 (continued)

**86. a.** $c(d) = 50 + 0.15d$, where $d \geq 0$

**b.** Interchange $c$ and $d$.

$$d = 50 + 0.15c$$
$$d - 50 = 0.15c$$
$$c = \frac{d - 50}{0.15}$$

$$\therefore\ c(d) = \frac{d - 50}{0.15}$$

**c.** In the inverse, $d$ represents the cost and $c$ represents the distance driven.

The inverse can be used to find the distance driven given the cost of the rental.

**d.** The domain of the inverse is $d \geq 50$, because the flat rate is \$50.

**89.** The function Chen used to calculate his answer is as follows:

$$f(x) = \frac{8(x+6) - 10}{2}$$

The variable $x$ represents the number Chen chose.

Chen chose $x = 11$.

$$\therefore f(11) = \frac{8(11+6) - 10}{2}$$
$$= \frac{8(17) - 10}{2}$$
$$= \frac{136 - 10}{2}$$
$$= 63$$

Marta used the inverse function to determine Chen's original number.

Switch the variables.

$$x = \frac{8(y+6)-10}{2}$$
$$2x = 8(y+6)-10$$
$$2x+10 = 8(y+6)$$
$$\frac{2x+10}{8} = y+6$$
$$y = \frac{2x+10}{8} - 6$$

$$\therefore f^{-1}(x) = \frac{2x+10}{8} - 6$$

In the inverse, $x$ is the answer Chen gave.

Marta now used $f^{-1}(x)$ to calculate his original value.

$$f^{-1}(63) = \frac{2(63)+10}{8} - 6$$
$$= \frac{126+10}{8} - 6$$
$$= \frac{136}{8} - 6$$
$$= 17 - 6$$
$$= 11$$

Chen's original number is 11.

**94. a.** In $d(t) = \sqrt[3]{830t^2}$, the restriction on the value of $t$ is $t \geq 0$ because the duration of the hurricane must be non-negative.

**b.** Interchange $d$ and $t$ to find the inverse.

$$t = \sqrt[3]{830d^2}$$
$$t^3 = 830d^2 \quad \longleftarrow \text{Cube both sides.}$$
$$d^2 = \frac{t^3}{830}$$
$$d = \sqrt{\frac{t^3}{830}}$$

Only the non-negative root may be used. Because $t \geq 0$ in the original function, $d \geq 0$ in the inverse.

## Section 2: Activity 3 (continued)

**c.** The inverse is a function because for a given value of $t$ there is exactly one value of $d$.

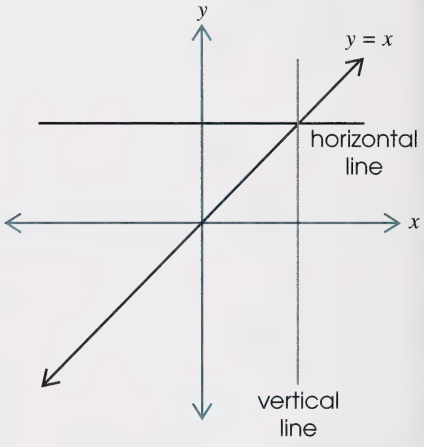
**d.** $d = \sqrt{\frac{100^3}{830}}$

$\doteq 35$

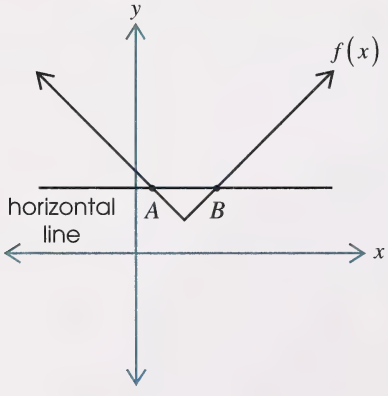
The hurricane lasts about 35 h.

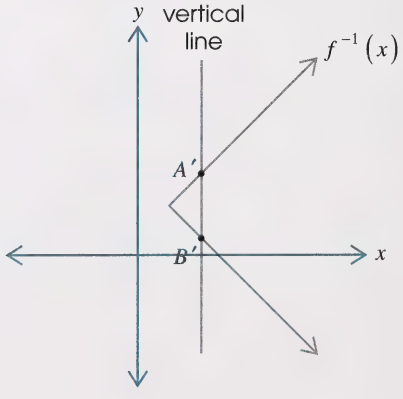
Be careful! In the inverse, $d$ represents time and $t$ represents the hurricane's diameter. Remember, you switched the variables.

**96. a.** A horizontal line becomes a vertical line when reflected in the line $y = x$.



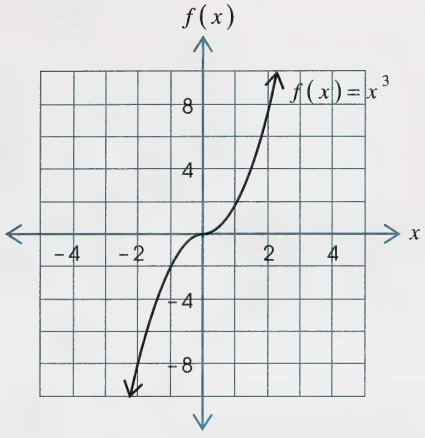

Suppose a horizontal line cuts a curve at two points: $A$ and $B$. When you draw inverses of both the function, $f$, and the horizontal line, you will see a vertical line crossing the inverse of $f$ at points $A'$ and $B'$, thus showing the inverse is not a function.




Therefore, if a horizontal line crosses the graph of a function at more than one point, the inverse of that function is not a function.

**b.**




Since every horizontal line crosses the graph of $y = x^3$ at exactly one point, the inverse of $y = x^3$ is a function.

**6.**

$$t = \frac{96^2 \times \text{mass}}{10\,378.64 \times \text{power}}$$

$$8 = \frac{96^2 \times 1500}{10\,378.64 \times \text{power}}$$

$$8 \times 10\,378.64 \times \text{power} = 96^2 \times 1500$$

$$\text{power} = \frac{96^2 \times 1500}{8 \times 10\,378.64}$$

$$\doteq 166.5 \text{ hp}$$

The required horsepower is approximately 166.5.

# Section 2: Follow-up Activities

## Extra Help

**1.** Try $f(5)$.

| Function, $f$ | Input | Multiply by 3 | | Add 4 | Output |
|---|---|---|---|---|---|
| $f(x)=3x+4$ | 5 | $\xrightarrow{\times 3}$ | 15 | $\xrightarrow{+4}$ | 19 |

Reverse the numbers and reverse the operations.

| Input | Subtract 4 | | Divide by 3 | Output | Inverse of $f$ |
|---|---|---|---|---|---|
| 19 | $\xrightarrow{-4}$ | 15 | $\xrightarrow{\div 3}$ | 5 | $f^{-1}(x)=\frac{x-4}{3}$ |

**2.** Try $f(3)$.

| Function, $f$ | Input | Cube | | Subtract 7 | | Divide by 4 | Output |
|---|---|---|---|---|---|---|---|
| $f(x)=\frac{x^3-7}{4}$ | 3 | $\xrightarrow{x^3}$ | 27 | $\xrightarrow{-7}$ | 20 | $\xrightarrow{\div 4}$ | 5 |

Reverse the numbers and reverse the operations.

| Input | Multiply by 4 | | Add 7 | | Take the Cube Root | Output | Inverse of $f$ |
|---|---|---|---|---|---|---|---|
| 5 | $\xrightarrow{\times 4}$ | 20 | $\xrightarrow{+7}$ | 27 | $\xrightarrow{\sqrt[3]{\ }}$ | 3 | $f^{-1}(x)=\sqrt[3]{4x+7}$ |

## Enrichment

**1. Property 1:** Closure

If $f$ and $g$ are rational functions, then $f+g$ is a rational function. For example, given $f(x)=\frac{x}{2}$ and $g(x)=\frac{3}{x}$,

$$(f+g)(x)=\frac{x}{2}+\frac{3}{x}$$
$$=\frac{x^2+6}{2x}$$

This is a rational function.

**Property 2:** Associative Property

If $f$, $g$, and $h$ are rational functions, then $f+(g+h)=(f+g)+h$. For example, given $f(x)=\frac{x}{2}$, $g(x)=x^2$, and $h(x)=x+2$, show that $f(x)+[g(x)+h(x)]=[f(x)+g(x)]+h(x)$.

| LS | RS |
|---|---|
| $f(x)+[g(x)+h(x)]$ | $[f(x)+g(x)]+h(x)$ |
| $=\frac{x}{2}+[x^2+(x+2)]$ | $=\left(\frac{x}{2}+x^2\right)+(x+2)$ |
| $=\frac{x}{2}+(x^2+x+2)$ | $=\frac{x+2x^2}{2}+(x+2)$ |
| $=\frac{x}{2}+\frac{2x^2+2x+4}{2}$ | $=\frac{x+2x^2}{2}+\frac{2x+4}{2}$ |
| $=\frac{2x^2+3x+4}{2}$ | $=\frac{2x^2+3x+4}{2}$ |

LS = RS

**Property 3**: Commutative Property

If $f$ and $g$ are rational functions, then $f+g=g+f$.

Given $f(x)=\frac{x}{x+1}$ and $g(x)=\frac{2}{x}$, show that $f(x)+g(x)=g(x)+f(x)$.

| LS | RS |
|---|---|
| $f(x)+g(x)$ | $g(x)+f(x)$ |
| $=\frac{x}{x+1}+\frac{2}{x}$ | $=\frac{2}{x}+\frac{x}{x+1}$ |
| $=\frac{x^2+2(x+1)}{x(x+1)}$ | $=\frac{2(x+1)+x^2}{x(x+1)}$ |
| $=\frac{x^2+2x+2}{x^2+x}$ | $=\frac{2x+2+x^2}{x^2+x}$ |
| | $=\frac{x^2+2x+2}{x^2+x}$ |

LS = RS

## Enrichment (continued)

**Property 4:** Zero Element

The zero element is $z(x)=0$, since for any rational function $f$, $f+z=z+f=f$.

| LS | RS |
|---|---|
| $(f+z)(x)$ | $f(x)$ |
| $=f(x)+z(x)$ | |
| $=f(x)+0$ | |
| $=f(x)$ | |

LS = RS

| LS | RS |
|---|---|
| $(z+f)(x)$ | $f(x)$ |
| $=z(x)+f(x)$ | |
| $=0+f(x)$ | |
| $=f(x)$ | |

LS = RS

**Property 5:** Additive Inverses

If $f$ is a rational function, then there exists a rational function $-f$, such that $f+(-f)=(-f+f)=z$.

For example, if $f(x)=\frac{x+2}{x-3}$, then

$$-f(x)=-\frac{x+2}{x-3}$$

Clearly, $f(x)+\left(-f(x)\right)=0=z(x)$.

| LS | RS |
|---|---|
| $f(x)+\left(-f(x)\right)$ | $z(x)$ |
| $=\frac{x+2}{x-3}+\left(-\frac{x+2}{x-3}\right)$ | $=0$ |
| $=0$ | |

LS = RS

Similarly, $-f(x)+f(x)=0$.

2. Subtraction is defined as adding additive inverses. For rational functions $f$ and $g$, $f-g$ means $f+(-g)$.

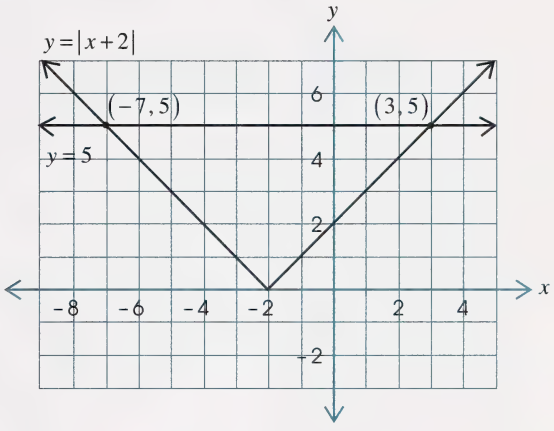
   Similarly, division is defined as multiplying by multiplicative inverses. For rational functions $f$ and $g$, $f\div g=f\bullet\frac{1}{g}$.

## Section 3: Activity 1

**1. a. Textbook questions a. and b. of "Explore: Interpret the Graphs," p. 288**

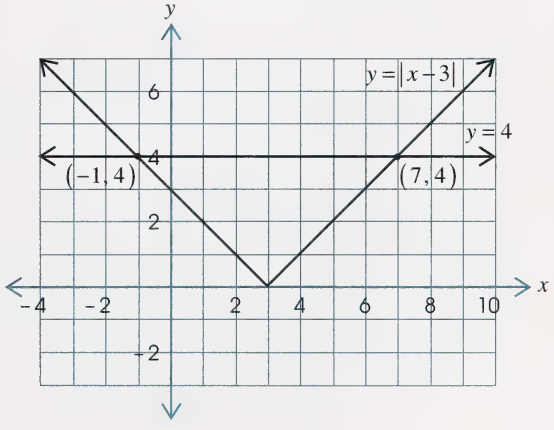
**a.** Graph $y = |x+2|$ and $y = 5$.

The graphs of $y = |x+2|$ and $y = 5$ intersect at $(-7, 5)$ and $(3, 5)$. Therefore, the solution to $|x+2| = 5$ is $x = -7$ or $x = 3$.




**b.** Graph $y = |x-3|$ and $y = 4$.

The graphs of $y = |x-3|$ and $y = 4$ intersect at $(-1, 4)$ and $(7, 4)$. Therefore, the solution to $|x-3| = 4$ is $x = -1$ or $x = 7$.




**b. Textbook questions 1 to 5 of "Inquire," p. 288**

**1. a.** You can verify your solutions by substituting them into the original equation.

## Section 3: Activity 1 (continued)

**b.** Verify that $x = -7$ and $x = 3$ satisfy $|x+2| = 5$.

For $x = -7$,

| LS | RS |
|---|---|
| $\|x+2\|$ | 5 |
| $=\|-7+2\|$ | |
| $=\|-5\|$ | |
| $=5$ | |

LS = RS

For $x = 3$,

| LS | RS |
|---|---|
| $\|x+2\|$ | 5 |
| $=\|3+2\|$ | |
| $=\|5\|$ | |
| $=5$ | |

LS = RS

The solution is $x = -7$ or $x = 3$.

Verify that $x = -1$ and $x = 7$ satisfy $|x-3| = 4$.

For $x = -1$,

| LS | RS |
|---|---|
| $\|x-3\|$ | 4 |
| $=\|-1-3\|$ | |
| $=\|-4\|$ | |
| $=4$ | |

LS = RS

For $x = 7$,

| LS | RS |
|---|---|
| $\|x-3\|$ | 4 |
| $=\|7-3\|$ | |
| $=\|4\|$ | |
| $=4$ | |

LS = RS

The solution is $x = -1$ or $x = 7$.

**2.** Each equation has two solutions because there are two points of intersection in each graph.

**3. a. Guess:** $x = 17$ or $x = -5$

**Check**

For $x = 17$,

| LS | RS |
|---|---|
| $\|x-6\|$ | 11 |
| $=\|17-6\|$ | |
| $=\|11\|$ | |
| $=11$ | |

LS = RS

For $x = -5$,

| LS | RS |
|---|---|
| $\|x-6\|$ | 11 |
| $=\|-5-6\|$ | |
| $=\|-11\|$ | |
| $=11$ | |

LS = RS

The solution is $x = 17$ or $x = -5$.

**b. Guess:** $x = -19$ or $x = 1$

**Check**

For $x = -19$,

| LS | RS |
|---|---|
| $\lvert x+9 \rvert$ | 10 |
| $= \lvert -19+9 \rvert$ | |
| $= \lvert -10 \rvert$ | |
| $= 10$ | |

LS = RS

For $x = 1$,

| LS | RS |
|---|---|
| $\lvert x+9 \rvert$ | 10 |
| $= \lvert 1+9 \rvert$ | |
| $= \lvert 10 \rvert$ | |
| $= 10$ | |

LS = RS

The solution is $x = -19$ or $x = 1$.

**c. Guess:** $x = -18$ or $x = -16$

**Check**

For $x = -18$,

| LS | RS |
|---|---|
| $\lvert x+17 \rvert$ | 1 |
| $= \lvert -18+17 \rvert$ | |
| $= \lvert -1 \rvert$ | |
| $= 1$ | |

LS = RS

For $x = -16$,

| LS | RS |
|---|---|
| $\lvert x+17 \rvert$ | 1 |
| $= \lvert -16+17 \rvert$ | |
| $= \lvert 1 \rvert$ | |
| $= 1$ | |

LS = RS

The solution is $x = -18$ or $x = -16$.

## Section 3: Activity 1 (continued)

**d.** **Guess:** $x = -5$ or $x = 11$

**Check**

For $x = -5$,

| LS | RS |
|---|---|
| $\lvert 3-x \rvert$ | 8 |
| $= \lvert 3-(-5) \rvert$ | |
| $= \lvert 3+5 \rvert$ | |
| $= 8$ | |

LS = RS

For $x = 11$,

| LS | RS |
|---|---|
| $\lvert 3-x \rvert$ | 8 |
| $= \lvert 3-11 \rvert$ | |
| $= \lvert -8 \rvert$ | |
| $= 8$ | |

LS = RS

The solution is $x = -5$ or $x = 11$.

**4.** **Guess:** $t = 365$ or $t = 355$

**Check**

For $t = 365$,

| LS | RS |
|---|---|
| $\lvert t-360 \rvert$ | 5 |
| $= \lvert 365-360 \rvert$ | |
| $= \lvert 5 \rvert$ | |
| $= 5$ | |

LS = RS

For $t = 355$,

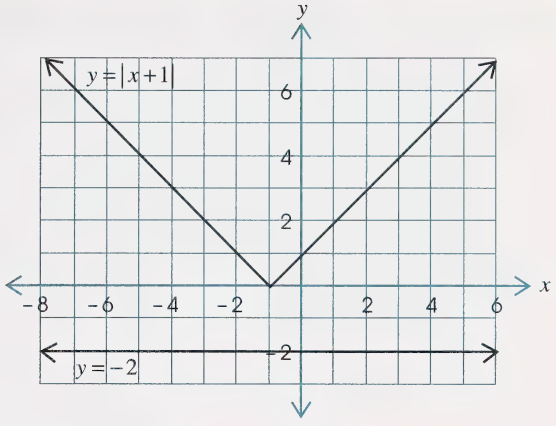
| LS | RS |
|---|---|
| $\lvert t-360 \rvert$ | 5 |
| $= \lvert 355-360 \rvert$ | |
| $= \lvert -5 \rvert$ | |
| $= 5$ | |

LS = RS

The maximum time is 365 min, and the minimum time is 355 min.

**5. a.** Graph $y = |x+1|$ and $y = -2$.

There is no solution because the graph of $y = -2$ does not intersect the graph of $y = |x+1|$.




**b.** The left side of the equation, $|x+1|$, can never equal the right side, $-2$, because absolute value is non-negative.

**2. Textbook questions 1, 8, 12, 20, 21, and 28 of "Practice," p. 296**

**1.** From the definition of absolute value, $|x| = 3$ means $x = 3$ or $-x = 3$.

Consider both cases.

$x = 3$ or $-x = 3$

$x = -3$

**Check**

For $x = 3$,

| LS | RS |
|---|---|
| $\|x\|$ | 3 |
| $= \|3\|$ | |
| $= 3$ | |

LS = RS

For $x = -3$,

| LS | RS |
|---|---|
| $\|x\|$ | 3 |
| $= \|-3\|$ | |
| $= 3$ | |

LS = RS

The solution is $x = 3$ or $x = -3$.

## Section 3: Activity 1 (continued)

**8.** Isolate the absolute value expression.

$$|2+m|-15=0$$
$$|2+m|=15$$

From the definition of absolute value, $|2+m|=15$ means $2+m=15$ or $-(2+m)=15$.

Consider both cases.

$\therefore\ 2+m=15$ or $-(2+m)=15$

$m=13$ $-2-m=15$

$-m=17$

$m=-17$

**Check**

For $m=13$,

| LS | RS |
|---|---|
| $\|2+m\|-15$ | 0 |
| $=\|2+13\|-15$ | |
| $=\|15\|-15$ | |
| $=15-15$ | |
| $=0$ | |

LS = RS

For $m=-17$,

| LS | RS |
|---|---|
| $\|2+m\|-15$ | 0 |
| $=\|2+(-17)\|-15$ | |
| $=\|-15\|-15$ | |
| $=15-15$ | |
| $=0$ | |

LS = RS

The solution is $m=13$ or $m=-17$.

**12.** From the definition of absolute value, $|3x+2|=8$ means $3x+2=8$ or $-(3x+2)=8$.

Consider both cases.

$$\begin{aligned} 3x+2&=8 \quad \text{or} \quad -(3x+2)=8\\ 3x&=6 \qquad\qquad -3x-2=8\\ x&=2 \qquad\qquad -3x=10\\ &\qquad\qquad x=-\frac{10}{3}\end{aligned}$$

**Check**

For $x=2$,

| LS | RS |
|---|---|
| $\|3x+2\|$ | 8 |
| $=\|3(2)+2\|$ | |
| $=\|6+2\|$ | |
| $=\|8\|$ | |
| $=8$ | |

LS = RS

For $x=-\frac{10}{3}$,

| LS | RS |
|---|---|
| $\|3x+2\|$ | 8 |
| $=\left\|3\left(-\frac{10}{3}\right)+2\right\|$ | |
| $=\|-10+2\|$ | |
| $=\|-8\|$ | |
| $=8$ | |

LS = RS

The solution is $x=2$ or $x=-\frac{10}{3}$.

**20.** From the definition of absolute value, $\left|\frac{2x+1}{3}\right|=5$ means $\frac{2x+1}{3}=5$ or $-\frac{2x+1}{3}=5$.

Consider both cases.

$$\begin{aligned} \frac{2x+1}{3}&=5 \quad \text{or} \quad -\frac{2x+1}{3}=5\\ 2x+1&=15 \qquad\qquad \frac{2x+1}{-3}=5\\ 2x&=14 \qquad\qquad 2x+1=-15\\ x&=7 \qquad\qquad 2x=-16\\ &\qquad\qquad x=-8\end{aligned}$$

## Section 3: Activity 1 (continued)

**Check**

For $x = 7$,

| LS | RS |
|---|---|
| $\left\lvert\frac{2x+1}{3}\right\rvert$ | 5 |
| $=\left\lvert\frac{2(7)+1}{3}\right\rvert$ | |
| $=\left\lvert\frac{14+1}{3}\right\rvert$ | |
| $=\left\lvert\frac{15}{3}\right\rvert$ | |
| $=\lvert 5\rvert$ | |
| $=5$ | |
| LS = RS | |

For $x = -8$,

| LS | RS |
|---|---|
| $\left\lvert\frac{2x+1}{3}\right\rvert$ | 5 |
| $=\left\lvert\frac{2(-8)+1}{3}\right\rvert$ | |
| $=\left\lvert\frac{-16+1}{3}\right\rvert$ | |
| $=\left\lvert-\frac{15}{3}\right\rvert$ | |
| $=\lvert -5\rvert$ | |
| $=5$ | |
| LS = RS | |

The solution is $x = 7$ or $x = -8$.

**21.** From the definition of absolute value, $|x| = x + 2$ means $x = x + 2$ or $-x = x + 2$.

Consider both cases.

$x = x + 2$ or $-x = x + 2$

$0 = 2$ $-2x = 2$

$x = -1$

Impossible!

**Check**

For $x = -1$,

| LS | RS |
|---|---|
| $\lvert x\rvert$ | $x + 2$ |
| $=\lvert -1\rvert$ | $=-1+2$ |
| $=1$ | $=1$ |
| LS = RS | |

The solution is $x = -1$.

**28.** From the definition of absolute value, $|1-2x| = x+2$ means $1-2x = x+2$ or $-(1-2x) = x+2$.

Consider both cases.

$$\begin{aligned} 1-2x &= x+2 \\ -2x &= x+1 \\ -3x &= 1 \\ x &= -\frac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} -(1-2x) &= x+2 \\ -1+2x &= x+2 \\ 2x &= x+3 \\ x &= 3 \end{aligned}$$

**Check**

For $x = -\frac{1}{3}$,

| LS | RS |
|---|---|
| $\lvert 1-2x \rvert$ | $x+2$ |
| $= \left\lvert 1-2\left(-\frac{1}{3}\right) \right\rvert$ | $= -\frac{1}{3}+2$ |
| $= \left\lvert 1+\frac{2}{3} \right\rvert$ | $= 1\frac{2}{3}$ |
| $= 1\frac{2}{3}$ | |

LS = RS

For $x = 3$,

| LS | RS |
|---|---|
| $\lvert 1-2x \rvert$ | $x+2$ |
| $= \lvert 1-2(3) \rvert$ | $= 3+2$ |
| $= \lvert 1-6 \rvert$ | $= 5$ |
| $= \lvert -5 \rvert$ | |
| $= 5$ | |

LS = RS

The solution is $x = -\frac{1}{3}$ or $x = 3$.

**3. Textbook questions 35, 38, 42, and 46 of "Practice," p. 296**

**35.** $|x+3| = |x-3|$ means

$$\begin{aligned} x+3 &= x-3 \\ x &= x-6 \\ 0 &= -6 \\ &\uparrow \\ &\text{Impossible!} \end{aligned} \quad \text{or} \quad \begin{aligned} x+3 &= -(x-3) \\ x+3 &= -x+3 \\ 2x+3 &= 3 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

## Section 3: Activity 1 (continued)

**Check**

For $x = 0$,

| LS | RS |
|---|---|
| $\lvert x+3\rvert$ | $\lvert x-3\rvert$ |
| $=\lvert 0+3\rvert$ | $=\lvert 0-3\rvert$ |
| $=\lvert 3\rvert$ | $=\lvert -3\rvert$ |
| $=3$ | $=3$ |

LS = RS

The solution is $x = 0$.

**38.** Rewrite the equation with one absolute value expression on each side of the equal sign.

$$\lvert x-6\rvert - \lvert 4x+6\rvert = 0$$
$$\lvert x-6\rvert = \lvert 4x+6\rvert$$

$\therefore$ $x-6 = 4x+6$ or $x-6 = -(4x+6)$

$x = 4x+12$ $\qquad$ $x-6 = -4x-6$

$-3x = 12$ $\qquad$ $x = -4x$

$x = -4$ $\qquad$ $5x = 0$

$x = 0$

**Check**

For $x = -4$,

| LS | RS |
|---|---|
| $\lvert x-6\rvert - \lvert 4x+6\rvert$ | 0 |
| $=\lvert -4-6\rvert - \lvert 4(-4)+6\rvert$ | |
| $=\lvert -10\rvert - \lvert -16+6\rvert$ | |
| $=\lvert -10\rvert - \lvert -10\rvert$ | |
| $=10-10$ | |
| $=0$ | |

LS = RS

For $x = 0$,

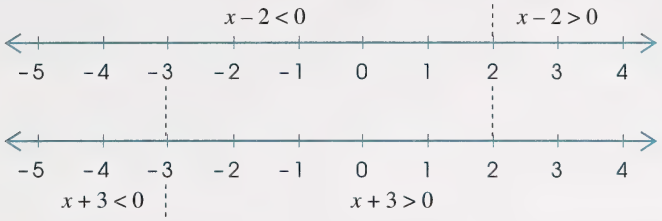
| LS | RS |
|---|---|
| $\lvert x-6\rvert - \lvert 4x+6\rvert$ | 0 |
| $=\lvert 0-6\rvert - \lvert 4(0)+6\rvert$ | |
| $=\lvert -6\rvert - \lvert 6\rvert$ | |
| $=0$ | |

LS = RS

The solution is $x = -4$ or $x = 0$.

**42.** $|x-2|+|x+3|=5$

To determine the intervals to be considered, graph the points for which $x-2$ and $x+3$ are positive or negative. The points 2 and $-3$ are the critical numbers.




There are three intervals to consider: $x<-3$, $-3<x<2$, and $x>2$.

When $x<-3$, both $x-2$ and $x+3$ are negative. In this interval, $|x-2|+|x+3|=5$ becomes

$$\begin{aligned} -(x-2)+[-(x+3)] &= 5 \\ -x+2-x-3 &= 5 \\ -2x-1 &= 5 \\ -2x &= 6 \\ x &= -3 \end{aligned}$$

This result is not true because $x<-3$ and $x=-3$ cannot both be true.

When $-3<x<2$, $x-2$ is negative and $x+3$ is positive.

In this interval, $|x-2|+|x+3|=5$ becomes

$$\begin{aligned} -(x-2)+x+3 &= 5 \\ -x+2+x+3 &= 5 \\ 5 &= 5 \end{aligned}$$

This result is always true. Therefore, $-3<x<2$ is part of the solution.

## Section 3: Activity 1 (continued)

When $x > 2$, $x-2$ and $x+3$ are positive.

In this interval, $|x-2|+|x+3|=5$ becomes

$$\begin{aligned} x-2+x+3&=5\\ 2x+1&=5\\ 2x&=4\\ x&=2 \end{aligned}$$

This result is not true because $x > 2$ and $x = 2$ cannot both be true.

Check the two critical numbers, $-3$ and 2, which have not been tested.

For $x=-3$,

| LS | RS |
|---|---|
| $\|x-2\|+\|x+3\|$ | 5 |
| $=\|-3-2\|+\|-3+3\|$ | |
| $=\|-5\|+\|0\|$ | |
| $=5$ | |

LS = RS

For $x=2$,

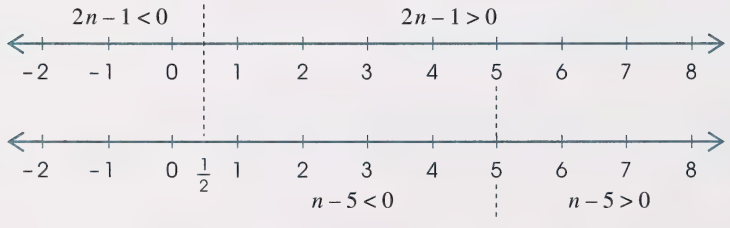
| LS | RS |
|---|---|
| $\|x-2\|+\|x+3\|$ | 5 |
| $=\|2-2\|+\|2+3\|$ | |
| $=\|0\|+\|5\|$ | |
| $=0+5$ | |
| $=5$ | |

LS = RS

The solution is $-3 \le x \le 2$.

**46.** $|2n-1|+|n-5|=8$

To determine the intervals to be considered, graph the points for which $2n-1$ and $n-5$ are positive or negative. The points $\frac{1}{2}$ and 5 are critical numbers.




There are three intervals to consider: $n < \frac{1}{2}$, $\frac{1}{2} < n < 5$, and $n > 5$.

When $n < \frac{1}{2}$, $2n-1$ and $n-5$ are negative.

In this interval, $|2n-1|+|n-5|=8$ becomes

$$\begin{aligned} -(2n-1)+[-(n-5)] &= 8 \\ -2n+1-n+5 &= 8 \\ -3n+6 &= 8 \\ -3n &= 2 \\ n &= -\frac{2}{3} \end{aligned}$$

When $\frac{1}{2} < n < 5$, $2n-1$ is positive and $n-5$ is negative.

In this interval, $|2n-1|+|n-5|=8$ becomes

$$\begin{aligned} (2n-1)+[-(n-5)] &= 8 \\ 2n-1-n+5 &= 8 \\ n+4 &= 8 \\ n &= 4 \end{aligned}$$

When $n > 5$, $2n-1$ and $n-5$ are positive.

In this interval, $|2n-1|+|n-5|=8$ becomes

$$\begin{aligned} (2n-1)+(n-5) &= 8 \\ 3n-6 &= 8 \\ 3n &= 14 \\ n &= \frac{14}{3} \\ &= 4\frac{2}{3} \end{aligned}$$

This result is not true because $n > 5$ and $n = 4\frac{2}{3}$ cannot both be true.

## Section 3: Activity 1 (continued)

Check $n=-\frac{2}{3}$ and $n=4$ and the critical values $\frac{1}{2}$ and 5.

For $n=-\frac{2}{3}$,

| LS | RS |
|---|---|
| $\lvert 2n-1\rvert+\lvert n-5\rvert$ | 8 |
| $=\left\lvert 2\left(-\frac{2}{3}\right)-1\right\rvert+\left\lvert -\frac{2}{3}-5\right\rvert$ | |
| $=\left\lvert -\frac{4}{3}-1\right\rvert+\left\lvert -5\frac{2}{3}\right\rvert$ | |
| $=2\frac{1}{3}+5\frac{2}{3}$ | |
| $=8$ | |

LS = RS

For $n=4$,

| LS | RS |
|---|---|
| $\lvert 2n-1\rvert+\lvert n-5\rvert$ | 8 |
| $=\lvert 2(4)-1\rvert+\lvert 4-5\rvert$ | |
| $=\lvert 8-1\rvert+\lvert -1\rvert$ | |
| $=7+1$ | |
| $=8$ | |

LS = RS

For $n=\frac{1}{2}$,

| LS | RS |
|---|---|
| $\lvert 2n-1\rvert+\lvert n-5\rvert$ | 8 |
| $=\left\lvert 2\left(\frac{1}{2}\right)-1\right\rvert+\left\lvert \frac{1}{2}-5\right\rvert$ | |
| $=\lvert 1-1\rvert+\left\lvert -4\frac{1}{2}\right\rvert$ | |
| $=0+4\frac{1}{2}$ | |
| $=4\frac{1}{2}$ | |

LS ≠ RS

For $n=5$,

| LS | RS |
|---|---|
| $\lvert 2n-1\rvert+\lvert n-5\rvert$ | 8 |
| $=\lvert 2(5)-1\rvert+\lvert 5-5\rvert$ | |
| $=\lvert 10-1\rvert+\lvert 0\rvert$ | |
| $=9+0$ | |
| $=9$ | |

LS ≠ RS

The solution is $n=-\frac{2}{3}$ or $n=4$.

**4. Textbook questions 49, 50, 56, 59, and 61 of "Practice," p. 296**

**49.** $|x|<2$

$\therefore\ -2<x<2$

So, $|x|<2$ when $-2<x<2$.

**50.** $|x| \geq 5$

$\therefore\ x \geq 5$ or $x \leq -5$

So, $|x| \geq 5$ when $x \geq 5$ or $x \leq -5$.

**56.** $|x-4| > 8$

$\therefore\ x-4 > 8$ or $x-4 < -8$

$x > 12$ $\quad$ $x < -4$

So, $|x-4| > 8$ when $x > 12$ or $x < -4$.

**59.** $|1-2x| > 5$

$\therefore\ 1-2x > 5$ or $1-2x < -5$

$-2x > 4$ $\quad$ $-2x < -6$

$x < -2$ $\quad$ $x > 3$

Recall that multiplying or dividing by a negative number changes the direction of the inequality.

So, $|1-2x| > 5$ when $x < -2$ or $x > 3$.

**61.** $\left|\frac{x}{2}+1\right| \leq 3$

$\therefore\ -3 \leq \frac{x}{2}+1 \leq 3$

$-4 \leq \frac{x}{2} \leq 2$ ← Subtract 1 from each part.

$-8 \leq x \leq 4$ ← Multiply each part by 2.

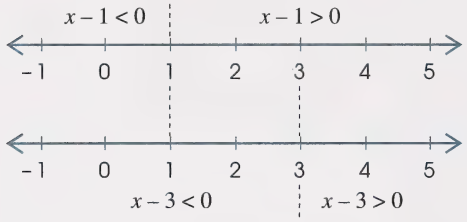
So, $\left|\frac{x}{2}+1\right| \leq 3$ when $-8 \leq x \leq 4$.

## Section 3: Activity 1 (continued)

**5. a. Textbook questions 63, 66, 73, 76, 83, and 91 of "Practice," p. 296**

**63.** $|x-1|<|x-3|$

To determine the intervals to be considered, graph the points for which $x-1$ and $x-3$ are positive or negative. The points 1 and 3 are the critical numbers.




There are three intervals to consider: $x<1$, $1<x<3$, and $x>3$.

When $x<1$, both $x-1$ and $x-3$ are negative.

In this interval, $|x-1|<|x-3|$ becomes

$$-(x-1)<-(x-3)$$
$$-x+1<-x+3$$
$$1<3$$

This result is true; so, in the interval $x<1$, all values of the variable satisfy the inequality.

When $1<x<3$, $x-1$ is positive and $x-3$ is negative.

In this interval, $|x-1|<|x-3|$ becomes

$$x-1<-(x-3)$$
$$x-1<-x+3$$
$$2x<4$$
$$x<2$$

In this interval, values of $x<2$ satisfy the inequality.

When $x>3$, both $x-1$ and $x-3$ are positive.

In this interval,

$$x-1<x-3$$
$$-1<-3$$

This result is not true; so, in this interval, there are no solutions.

Check the two critical numbers, 1 and 3, which have not been tested.

For $x=1$,

| LS | RS |
|---|---|
| $\lvert x-1\rvert$ | $\lvert x-3\rvert$ |
| $=\lvert 1-1\rvert$ | $=\lvert 1-3\rvert$ |
| $=\lvert 0\rvert$ | $=\lvert -2\rvert$ |
| $=0$ | $=2$ |

LS $<$ RS

For $x=3$,

| LS | RS |
|---|---|
| $\lvert x-1\rvert$ | $\lvert x-3\rvert$ |
| $=\lvert 3-1\rvert$ | $=\lvert 3-3\rvert$ |
| $=2$ | $=0$ |

LS $\not<$ RS

The solution includes 1 and does not include 3.

The solution is $x<2$.

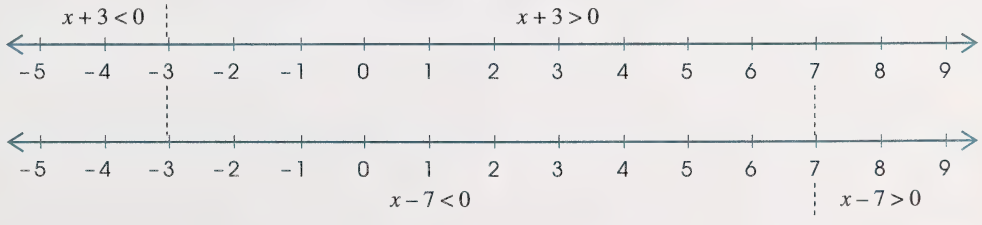
*Note: This includes the first interval and $x=1$.*

**66.** $|x-7|-|x+3|\geq 0$

Rewrite the inequality with one absolute value symbol on each side of the inequality.

$$|x-7|\geq|x+3|$$

To determine the intervals to be considered, graph the points for which $x-7$ and $x+3$ are positive or negative. The points 7 and $-3$ are the critical numbers.



## Section 3: Activity 1 (continued)

There are three intervals to consider: $x < -3$, $-3 < x < 7$, and $x > 7$.

When $x < -3$, $x+3$ and $x-7$ are negative.

In this interval, $|x-7| \geq |x+3|$ becomes

$$\begin{aligned} -(x-7) &\geq -(x+3) \\ -x+7 &\geq -x-3 \\ 7 &\geq -3 \end{aligned}$$

This result is true; so, in the interval $x < -3$, all values of the variable satisfy the inequality.

When $-3 < x < 7$, $x+3$ is positive and $x-7$ is negative.

In this interval, $|x-7| \geq |x+3|$ becomes

$$\begin{aligned} -(x-7) &\geq x+3 \\ -x+7 &\geq x+3 \\ 4 &\geq 2x \\ 2x &\leq 4 \\ x &\leq 2 \end{aligned}$$

In this interval, all values of $x \leq 2$ satisfy the inequality.

When $x > 7$, both $x+3$ and $x-7$ are positive.

In this interval,

$$\begin{aligned} |x-7| &\geq |x+3| \\ x-7 &\geq x+3 \\ -7 &\geq 3 \end{aligned}$$

This result is not true; so in this interval there are no solutions.

Check the two critical numbers, $-3$ and 7, which have not been tested.

For $x=-3$,

| LS | RS |
|---|---|
| $\lvert x-7\rvert-\lvert x+3\rvert$ | 0 |
| $=\lvert -3-7\rvert-\lvert -3+3\rvert$ | |
| $=10-0$ | |
| $=10$ | |

LS $\geq$ RS

For $x=7$,

| LS | RS |
|---|---|
| $\lvert x-7\rvert-\lvert x+3\rvert$ | 0 |
| $=\lvert 7-7\rvert-\lvert 7+3\rvert$ | |
| $=0-10$ | |
| $=-10$ | |

LS $\geq$ RS

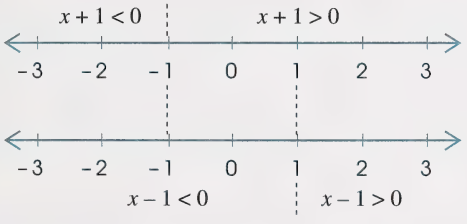
The solution includes $-3$ and does not include 7.

So, the solution is $x \leq 2$.

*Note: This includes the first interval and $x=-3$.*

**73.** $|x+1|+|x-1| \leq 6$

To determine the intervals to be considered, graph the points for which $x+1$ and $x-1$ are positive or negative. The points $-1$ and 1 are the critical numbers.




There are three intervals to consider: $x<-1$, $-1<x<1$, and $x>1$.

When $x<-1$, both $x+1$ and $x-1$ are negative.

## Section 3: Activity 1 (continued)

In this interval, $|x+1|+|x-1| \leq 6$ becomes

$$\begin{aligned} -(x+1)+[-(x-1)] &\leq 6 \\ -x-1-x+1 &\leq 6 \\ -2x &\leq 6 \\ x &\geq -3 \end{aligned}$$

In this interval, values of $x \geq -3$ satisfy the inequality.

When $-1 < x < 1$, $x+1$ is positive and $x-1$ is negative.

In this interval, $|x+1|+|x-1| \leq 6$ becomes

$$\begin{aligned} (x+1)+[-(x-1)] &\leq 6 \\ x+1-x+1 &\leq 6 \\ 2 &\leq 6 \end{aligned}$$

This result is true; so all values of $x$ in this interval satisfy the inequality.

When $x > 1$, both $x+1$ and $x-1$ are positive.

In this interval, $|x+1|+|x-1| \leq 6$ becomes

$$\begin{aligned} (x+1)+(x-1) &\leq 6 \\ 2x &\leq 6 \\ x &\leq 3 \end{aligned}$$

In this interval, values of $x \leq 3$ satisfy the inequality.

Check the two critical numbers, $\pm 1$, which have not been tested.

For $x = -1$,

| LS | RS |
|---|---|
| $\lvert x+1\rvert+\lvert x-1\rvert$ | 6 |
| $=\lvert -1+1\rvert+\lvert -1-1\rvert$ | |
| $=0+\lvert -2\rvert$ | |
| $=2$ | |

LS $\leq$ RS

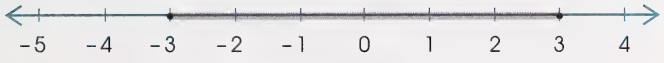
For $x = 1$,

| LS | RS |
|---|---|
| $\lvert x+1\rvert+\lvert x-1\rvert$ | 6 |
| $=\lvert 1+1\rvert+\lvert 1-1\rvert$ | |
| $=2+0$ | |
| $=2$ | |

LS $\leq$ RS

The solution includes $-1$ and 1.

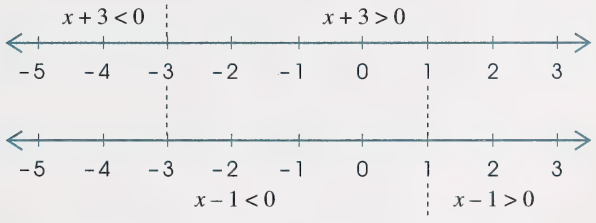
The solution is $-3 \leq x \leq 3$.




*Note: This includes +1 and −1 and all of the second interval.*

**76.** $\lvert x+3\rvert+\lvert x-1\rvert > 10$

To determine the intervals to be considered, graph the points for which $x+3$ and $x-1$ are positive or negative. The points $-3$ and 1 are the critical numbers.




There are three intervals to consider: $x < -3$, $-3 < x < 1$, and $x > 1$.

## Section 3: Activity 1 (continued)

When $x < -3$, both $x+3$ and $x-1$ are negative.

In this interval, $|x+3|+|x-1|>10$ becomes

$$\begin{aligned} -(x+3)+[-(x-1)] &> 10 \\ -x-3-x+1 &> 10 \\ -2x-2 &> 10 \\ -2x &> 12 \\ x &< -6 \end{aligned}$$

In this interval, values of $x < -6$ satisfy the inequality.

When $-3 < x < 1$, $x+3$ is positive and $x-1$ is negative.

In this interval, $|x+3|+|x-1|>10$ becomes

$$\begin{aligned} (x+3)+[-(x-1)] &> 10 \\ x+3-x+1 &> 10 \\ 4 &> 10 \end{aligned}$$

This result is not true; so, in this interval, there are no solutions.

When $x > 1$, both $x+3$ and $x-1$ are positive.

In this interval, $|x+3|+|x-1|>10$ becomes

$$\begin{aligned} (x+3)+(x-1) &> 10 \\ 2x+2 &> 10 \\ 2x &> 8 \\ x &> 4 \end{aligned}$$

In this interval, values of $x > 4$ satisfy the inequality.

Check the two critical numbers, $-3$ and 1, which have not been tested.

For $x = -3$,

| LS | RS |
|---|---|
| $\lvert x+3\rvert+\lvert x-1\rvert$ | 10 |
| $=\lvert -3+3\rvert+\lvert -3-1\rvert$ | |
| $=0+\lvert -4\rvert$ | |
| $=4$ | |

LS $\not>$ RS

For $x = 1$,

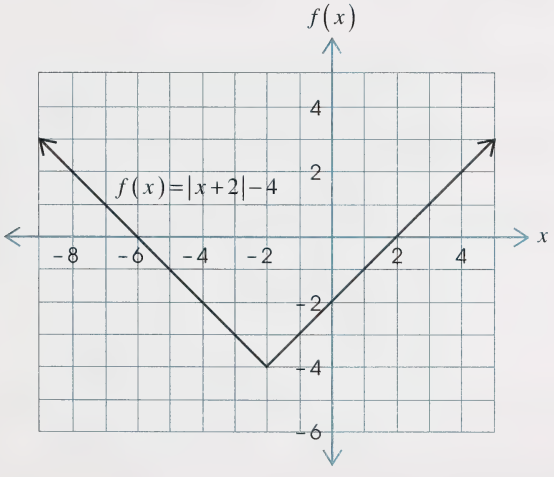
| LS | RS |
|---|---|
| $\lvert x+3\rvert+\lvert x-1\rvert$ | 10 |
| $=\lvert 1+3\rvert+\lvert 1-1\rvert$ | |
| $=4+0$ | |
| $=4$ | |

LS $\not>$ RS

The solution does not include $-3$ or 1.

The solution is $x < -6$ or $x > 4$.

**83.**




**a.** The domain is the set of reals, and the range is $f(x) \geq -4$.

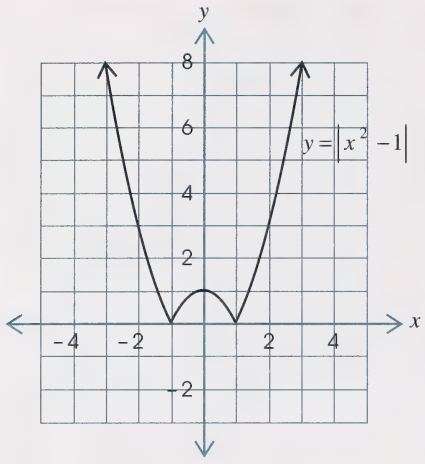
**b.** The zeros are $-6$ and 2.

**c.** $f(x) \geq 0$ when $x \leq -6$ or $x \geq 2$

**d.** The graph is symmetric with respect to $x = -2$.

## Section 3: Activity 1 (continued)

**91.**




**a.** The domain is the set of reals, and the range is $y \geq 0$.

**b.** The zeros are $\pm 1$.

**c.** The values of $x$ for which $y > 0$ are all real numbers except $\pm 1$.

**d.** The graph is symmetric about the $y$-axis.

**b. Textbook questions 99, 103, and 111.a. of "Applications and Problem Solving," p. 297**

**99.** Solve $|t - 360| \leq 20$.

$$\therefore \; -20 \leq t - 360 \leq 20$$
$$-20 + 360 \leq t - 360 + 360 \leq 20 + 360$$
$$340 \leq t \leq 380$$

The launch window is between 340 min and 380 min from 01:00.

Now, 340 min = 5 h 40 min, and 380 min = 6 h 20 min.

Therefore, the launch window is between 06:40 and 07:20.

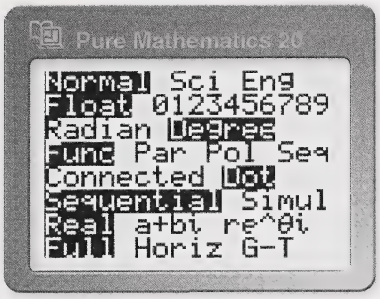
**103.** Solve $|T - 50| = 50$.

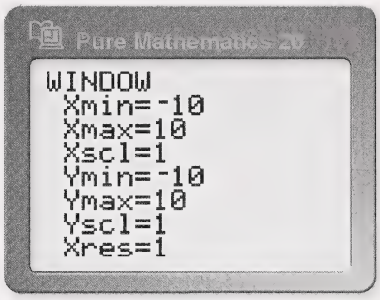
Apply the definition of absolute value.

$$T - 50 = 50 \quad \text{or} \quad -(T - 50) = 50$$
$$T = 100 \qquad -T + 50 = 50$$
$$-T = 0$$
$$T = 0$$

This compound is a liquid between 0°C and 100°C; therefore, it is water.

**111. a.** Ensure the calculator is in Dot mode and you are using the standard window settings.



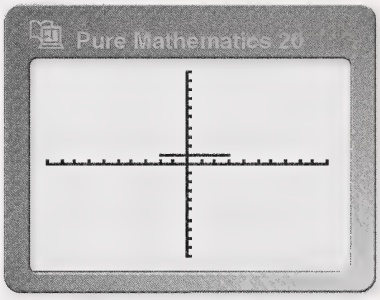

Graph $y=|x+2|+|x-3|=5$ on your graphing calculator.

Y= MATH [ Select the NUM menu. ] 1

X,T,Θ,n + 2 ) + MATH

[ Select the NUM menu. ] 1 X,T,Θ,n −

3 ) 2nd [ TEST ] 1 5 GRAPH




The graph extends from $(-2,1)$ to $(3,1)$.

Therefore, the solution is $-2 \le x \le 3$.

**6. Textbook question 112 of "Applications and Problem Solving," p. 298**

**112. a.** $|x-1| \le 4$

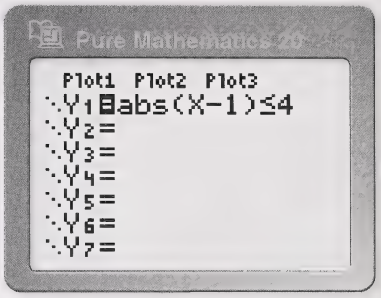
$-4 \le x-1 \le 4$ ← Add 1 to each part.

$-3 \le x \le 5$

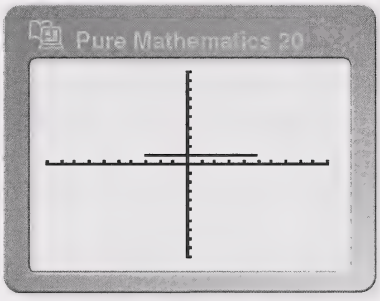
The solution is $-3 \le x \le 5$.

## Section 3: Activity 1 (continued)

**b.** Y= MATH [ Select the NUM menu. ] 1 X,T,Θ,*n* (−) 1 ) 2nd [ TEST ] 6 4




**c.** After entering the relation, press GRAPH.




**d.** The calculator correctly graphs $-3 \le x \le 5$ on the line $y = 1$.

**e.** If the inequality is ≤, the endpoints are included. If the inequality is <, the endpoints are not included. **Note:** In the textbook, this question was labelled as 112.d. with a pencil icon in front of it.

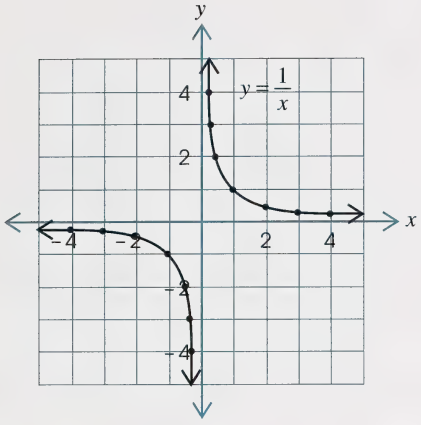
## Section 3: Activity 2

**1. a. Textbook questions a. and b. of "Explore: Draw a Graph," p. 299**

**a.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 2 | 3 | 4 |
| $y$ | 4 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ |
| $x$ | $-\frac{1}{4}$ | $-\frac{1}{3}$ | $-\frac{1}{2}$ | −1 | −2 | −3 | −4 |
| $y$ | −4 | −3 | −2 | −1 | $-\frac{1}{2}$ | $-\frac{1}{3}$ | $-\frac{1}{4}$ |

**b.**




**b. Textbook questions 1 to 6 of "Inquire," pp. 299 and 300**

**1.** The graph of $y = \frac{1}{x}$ appears only in the first and third quadrants, because $x$ and $y$ have the same sign.

**2.** The domain is $x \neq 0$.

**3.** **Table 1**

| | | | | |
|---|---|---|---|---|
| **x** | 1 | 10 | 100 | 1000 |
| **y** | 1 | 0.1 | 0.01 | 0.001 |
| **x** | 1 | 0.1 | 0.01 | 0.001 |
| **y** | 1 | 10 | 100 | 1000 |

**Table 2**

| | | | | |
|---|---|---|---|---|
| **x** | −1 | −10 | −100 | −1000 |
| **y** | −1 | −0.1 | −0.01 | −0.001 |
| **x** | −1 | −0.1 | −0.01 | −0.001 |
| **y** | −1 | −10 | −100 | −1000 |

**4.** **a.** As $x$ increases, $y$ approaches 0.

**b.** The value of $y$ will never reach zero because $xy = 1$.

**c.** As $x$ decreases, $y$ increases.

**d.** The value of $y$ will never reach a maximum. For any value of $y$, however large, there is a value of $x$, such that $x = \frac{1}{y}$.

## Section 3: Activity 2 (continued)

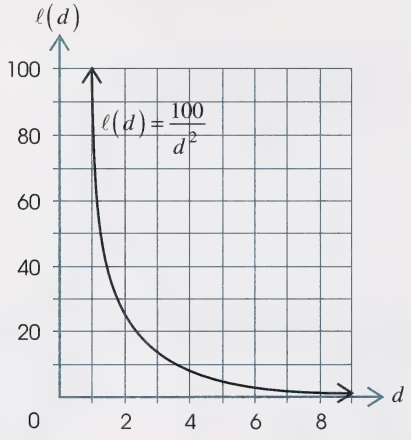
**5. a.** As $x$ decreases, $y$ approaches 0.

**b.** The value of $y$ will never reach 0, since $xy = 1$.

**c.** As $x$ increases, $y$ decreases in value.

**d.** The value of $y$ will never reach a minimum. For any value of $y$, however small, there is a value of $x$ such that $x = \frac{1}{y}$.

**6. a.** The domain is $d > 0$, and the range is $\ell(d) > 0$.




**b.** $\ell(d) = \frac{100}{d^2}$

$\ell(5) = \frac{100}{5^2}$

$= \frac{100}{25}$

$= 4$

The illuminance at a distance of 5 m from the light is $4\ \ell x$.

**c.** $\ell(d) = \dfrac{100}{d^2}$

$$25 = \frac{100}{d^2}$$

$$25d^2 = 100$$

$$d^2 = 4$$

$$d = 2$$

The illuminance is $25\,\ell x$ at a distance of 2 m from the light.

**2. a. Textbook questions 1, 2, 4, 8 to 12, 15, 21, 27, 37, and 46 of "Practice," pp. 308 and 309**

**1.** The function $y = x^5 + 2x^3 - 0.6$ is a polynomial function. It is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where the coefficients $a_n$, $a_{n-1}$, $a_{n-2}$, ..., $a_0$ represent real numbers, where $a_n \neq 0$, and the exponents are whole numbers.

**2.** The function $y = \frac{4}{x-2}$ is a rational function. It is the quotient of two polynomials.

**4.** The function $y = |x - 6| + 2$ is neither a polynomial nor a rational function. The variable appears within an absolute value symbol. This function is an example of an absolute value function.

**8.** The function $y = 3^x + 4$ is neither a polynomial nor a rational function. The variable appears as an exponent. This function is an example of an exponential function.

**9.** This graph is a graph of a rational function. It is discontinuous at $x = -3$, the asymptote of the curve.

**10.** The graph could be the graph of a fourth-degree polynomial function. It is continuous and has 4 zeros.

**11.** The graph could be the graph of a rational function. It is discontinuous at $x = 2$ and $x = 4$, which appear to be vertical asymptotes. The $x$-axis appears to be a horizontal asymptote.

**12.** The graph could be the graph of a fifth-degree polynomial function. It is continuous and appears to have zeros at $x = -2$, 0, and 2.

## Section 3: Activity 2 (continued)

**15.** The function $g(x)=\frac{1}{x-3}$ is not defined when

$$x-3=0$$
$$x=3$$

Therefore, there is a vertical asymptote at $x=3$.

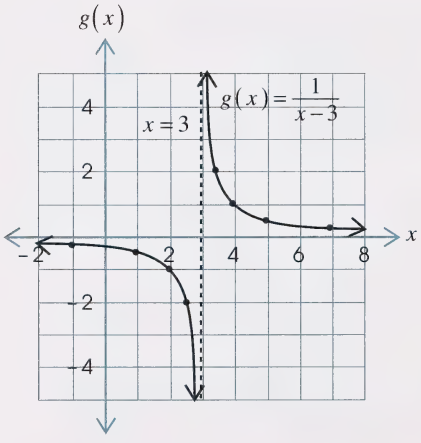
Determine the coordinates of points on both sides of $x=3$ to find the shape of the graph; then draw the graph.

### Right Side of Asymptote

| $x$ | $g(x)$ |
|---|---|
| 3.2 | 5 |
| 3.5 | 2 |
| 4 | 1 |
| 5 | 0.5 |
| 7 | 0.25 |

### Left Side of Asymptote

| $x$ | $g(x)$ |
|---|---|
| 2.8 | −5 |
| 2.5 | −2 |
| 2 | −1 |
| 1 | −0.5 |
| −1 | −0.25 |




**a.** The vertical asymptote is $x=3$, and the horizontal asymptote is $g(x)=0$.

**b.** The domain is $x\neq 3$, and the range is $g(x)\neq 0$. **Note:** The domain $x\neq 3$ implies all real numbers except $x=3$. Likewise, the range $g(x)\neq 0$ implies all real numbers except $g(x)=0$.

**21.** The function $f(x)=\frac{x}{x+2}$ is not defined when

$$x+2=0$$
$$x=-2$$

Therefore, there is a vertical asymptote at $x=-2$.

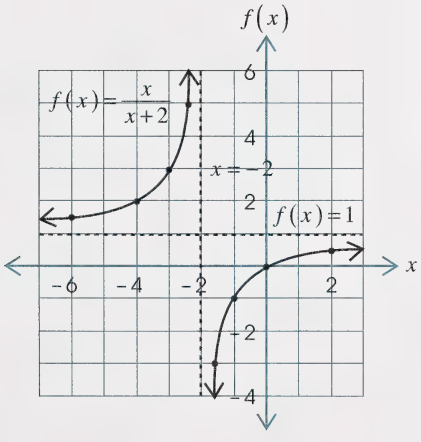
Determine the coordinates of points on both sides of $x=-2$ to find the shape of the graph; then draw the graph.

**Right Side of Asymptote**

| $x$ | $f(x)$ |
|---|---|
| −1.8 | −9 |
| −1.5 | −3 |
| −1 | −1 |
| 0 | 0 |
| 2 | 0.5 |
| 3 | 0.6 |

**Left Side of Asymptote**

| $x$ | $f(x)$ |
|---|---|
| −2.4 | 6 |
| −2.5 | 5 |
| −3 | 3 |
| −4 | 2 |
| −6 | 1.5 |
| −7 | 1.4 |




**a.** The vertical asymptote is $x=-2$, and the horizontal asymptote is $f(x)=1$.

**b.** The domain is $x\neq -2$, and the range is $f(x)\neq 1$.

## Section 3: Activity 2 (continued)

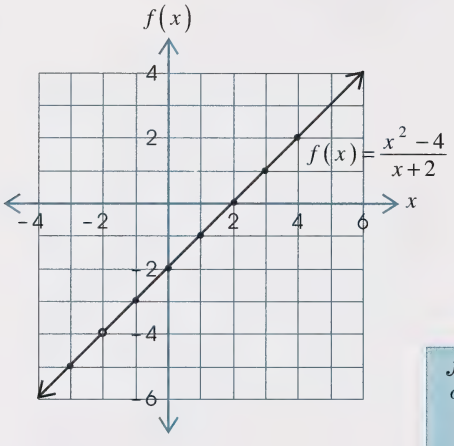
**27.** The function $f(x)=\frac{x^2-4}{x+2}$ is not defined when

$$x+2=0$$
$$x=-2$$

$$\therefore \frac{x^2-4}{x+2}=\frac{(x-2)(x+2)}{x+2}$$
$$=x-2, \text{ where } x\neq -2$$

Set up a table of values; then graph the curve.

| $x$ | $f(x)$ |
|---|---|
| $-3$ | $-5$ |
| $-2.1$ | $-4.1$ |
| $-2$ | undefined |
| $-1.9$ | $-3.9$ |
| $-1$ | $-3$ |
| $0$ | $-2$ |
| $1$ | $-1$ |
| $2$ | $0$ |
| $3$ | $1$ |
| $4$ | $2$ |




*Note: Place an open circle at $(-2,-4)$ to show the point of discontinuity.*

The domain is $x\neq -2$, and the range is $f(x)\neq -4$.

**37. a.** $y = \dfrac{x}{x-3}$

Switch $x$ and $y$ to find the inverse, and solve for $y$.

$$x = \frac{y}{y-3}$$
$$x(y-3) = y$$
$$xy - 3x = y$$
$$xy - y = 3x$$
$$y(x-1) = 3x$$
$$y = \frac{3x}{x-1}$$

$$\therefore f^{-1}(x) = \frac{3x}{x-1}$$

The inverse, $f^{-1}(x) = \frac{3x}{x-1}$, is not defined when

$$x - 1 = 0$$
$$x = 1$$

There is a vertical asymptote at $x = 1$.

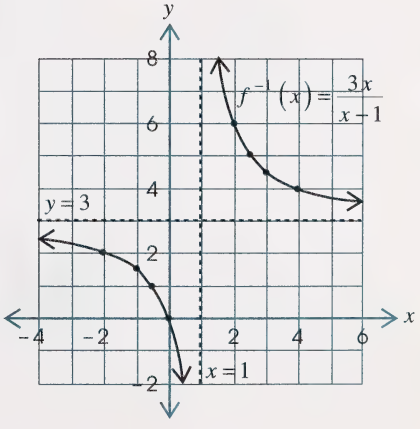
Determine the coordinates of points on both sides of $x = 1$ to find the shape of the graph; then draw the graph.

### Right Side of Asymptote

| $x$ | 1.1 | 1.5 | 2 | 2.5 | 3 | 4 |
|---|---|---|---|---|---|---|
| $y$ | 33 | 9 | 6 | 5 | 4.5 | 4 |

### Left Side of Asymptote

| $x$ | 0.9 | 0 | −0.5 | −1 | −2 | −4 |
|---|---|---|---|---|---|---|
| $y$ | −27 | 0 | 1 | 1.5 | 2 | 2.4 |

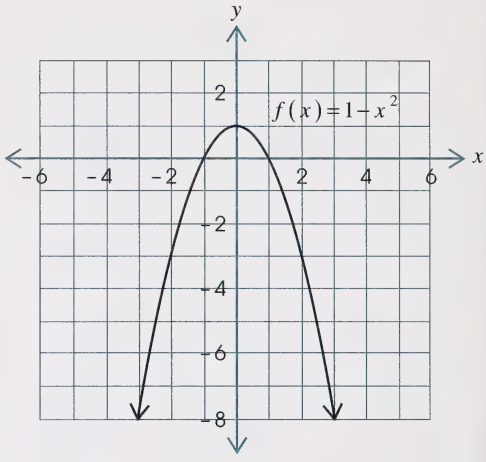



The vertical asymptote is $x = 1$, and the horizontal asymptote is $y = 3$.

## Section 3: Activity 2 (continued)

**b.** The domain is the set of real numbers where $x \neq 1$, and the range is the set of real numbers where $y \neq 3$.

**46. a.** Graph $f(x) = 1 - x^2$.




The zeros of $y = f(x)$ are $-1$ and $+1$, because

$$\begin{aligned} f(x) &= 1 - x^2 \\ f(\pm 1) &= 1 - (\pm 1)^2 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

**b.** The function $g(x) = \dfrac{1}{1 - x^2}$ is not defined when

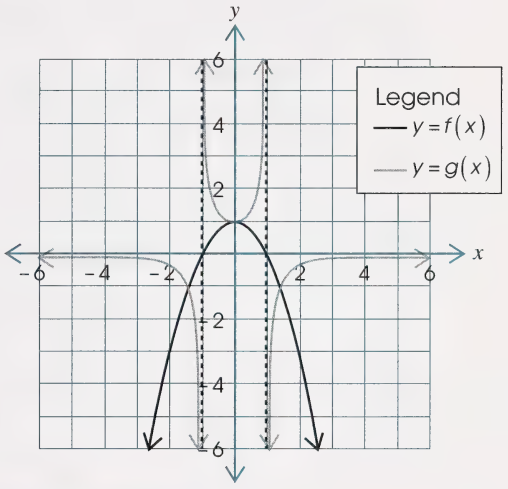
$$\begin{aligned} 1 - x^2 &= 0 \\ 1 &= x^2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

*Note: The zeros of $f(x)$ are the asymptotes of $g(x)$.*

The vertical asymptotes are $x = 1$ and $x = -1$.

**c.** Draw the asymptotes; then use a table of values to graph $g(x)=\frac{1}{1-x^2}$.

| $x$ | $g(x)$ |
|---|---|
| 0 | 1 |
| $\pm\frac{1}{2}$ | $\frac{4}{3}\doteq 1.3$ |
| $\pm\frac{9}{10}$ | $\frac{100}{19}\doteq 5.3$ |
| $\pm\frac{11}{10}$ | $-\frac{100}{21}\doteq -4.8$ |
| $\pm 2$ | $-\frac{1}{3}\doteq -0.3$ |
| $\pm 3$ | $-\frac{1}{8}\doteq -0.1$ |
| $\pm 4$ | $-\frac{1}{15}\doteq -0.07$ |




| | $f(x)=1-x^2$ | $g(x)=\frac{1}{1-x^2}$ |
|---|---|---|
| **Domain** | all real numbers | $x\neq\pm 1$ |
| **Range** | $y\leq 1$ | $y\geq 1$ or $y<0$ |
| **Symmetry** | about the $y$-axis | about the $y$-axis |
| **Zeros** | $\pm 1$ | none |
| **Asymptotes** | none | $x=\pm 1$ and $y=0$ |

## Section 3: Activity 2 (continued)

**b. Textbook question 97 of "Applications and Problem Solving," p. 310**

**97. a.** The function $f(x)=\dfrac{1}{x^2+x-6}$ is undefined when

$$x^2+x-6=0$$
$$(x+3)(x-2)=0$$

$$\therefore\ x+3=0 \quad \text{or} \quad x-2=0$$
$$x=-3 \qquad\qquad x=2$$

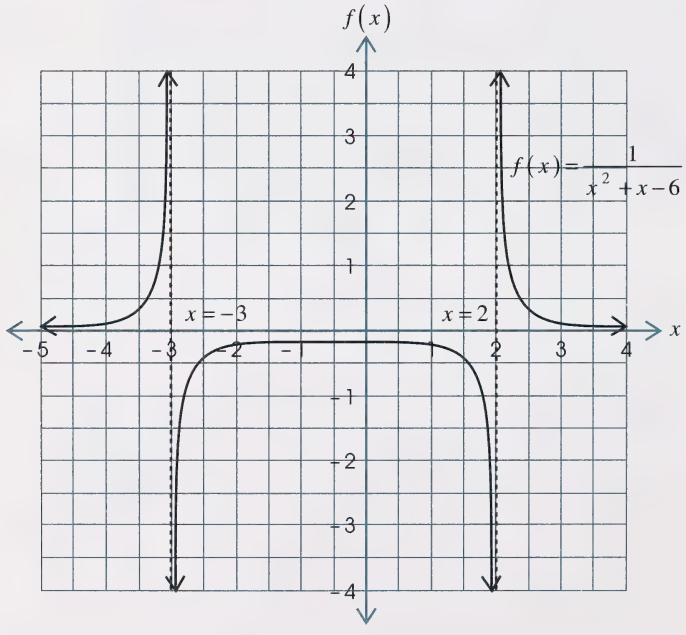
The vertical asymptotes of the graph are $x=-3$ and $x=2$.

The horizontal asymptote is $f(x)=0$.

Draw the asymptotes first; then use a table of values to complete the graph. The $f(x)$-values are correct to 2 decimal places.

| $x$ | $f(x)$ |
|---|---|
| −5 | 0.07 |
| −4 | 0.17 |
| −3.5 | 0.36 |
| −3.1 | 1.96 |
| −3.05 | 3.96 |
| −3 | undefined |
| −2.95 | −4.04 |
| −2.5 | −0.44 |
| −2 | −0.25 |
| −1 | 0.17 |

| $x$ | $f(x)$ |
|---|---|
| 0 | −0.17 |
| 1 | −0.25 |
| 1.9 | −2.04 |
| 1.95 | −4.04 |
| 2.0 | undefined |
| 2.05 | 3.96 |
| 2.5 | 0.36 |
| 3 | 0.17 |
| 4 | 0.07 |
| 5 | 0.04 |



**b.** The domain is $x \neq -3, 2$; and the range is $f(x) \neq 0$.

**3. a. Textbook questions 49, 54, 62, 69, 74, 77, 82, and 95 of "Practice," pp. 309 and 310**

**49.**

$$\frac{2}{x} = \frac{x+1}{3}$$

$$3x\left(\frac{2}{x}\right) = 3x\left(\frac{x+1}{3}\right)$$ ← Multiply both sides by the LCD, $3x$.

$$6 = x^2 + x$$

$$0 = x^2 + x - 6$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$\therefore\ x+3=0$ or $x-2=0$

$x=-3$ $\qquad$ $x=2$

**Check**

For $x=-3$,

| LS | RS |
|---|---|
| $\frac{2}{x}$ | $\frac{x+1}{3}$ |
| $=\frac{2}{-3}$ | $=\frac{-3+1}{3}$ |
| $=-\frac{2}{3}$ | $=-\frac{2}{3}$ |

LS = RS

For $x=2$,

| LS | RS |
|---|---|
| $\frac{2}{x}$ | $\frac{x+1}{3}$ |
| $=\frac{2}{2}$ | $=\frac{2+1}{3}$ |
| $=1$ | $=1$ |

LS = RS

The solution is $x=-3$ or $x=2$.

**54.**

$$\frac{2x}{x-1} = \frac{3x}{2x-2}$$

$$\frac{2x}{x-1} = \frac{3x}{2(x-1)}$$ ← Factor $2x-2$.

$$2(x-1)\times\frac{2x}{x-1} = 2(x-1)\times\frac{3x}{2(x-1)}$$ ← Multiply both sides by the LCD, $2(x-1)$.

$$4x = 3x$$

$$4x - 3x = 3x - 3x$$ ← Subtract $3x$ from both sides.

$$x = 0$$

## Section 3: Activity 2 (continued)

**Check**

| LS | RS |
|---|---|
| $\frac{2x}{x-1}$ | $\frac{3x}{2x-2}$ |
| $=\frac{2(0)}{0-1}$ | $=\frac{3(0)}{2(0)-2}$ |
| $=0$ | $=0$ |

LS = RS

The solution is $x=0$.

**62.**

$$\frac{1}{x-2}=\frac{x+1}{3x-6}-\frac{5x}{6}$$

$$\frac{1}{x-2}=\frac{x+1}{3(x-2)}-\frac{5x}{6}$$ ← Factor $3x-6$.

$$6(x-2)\frac{1}{x-2}=6(x-2)\left[\frac{x+1}{3(x-2)}-\frac{5x}{6}\right]$$ ← Multiply both sides by the LCD, $6(x-2)$.

$$6=2(x+1)-5x(x-2)$$

$$6=2x+2-5x^2+10x$$

$$6=-5x^2+12x+2$$

$$5x^2-12x+4=0$$

$$(5x-2)(x-2)=0$$

$\therefore\ 5x-2=0$ or $x-2=0$

$5x=2$ $\quad x=2$

$x=\frac{2}{5}$

When $x=2$, $\frac{1}{x-2}$ and $\frac{x+1}{3x-6}$ are undefined; therefore $x\neq 2$.

**Check**

For $x=\frac{2}{5}$,

| LS | RS |
|---|---|
| $\frac{1}{x-2}$ | $\frac{x+1}{3x-6}-\frac{5x}{6}$ |
| $=\frac{1}{\frac{2}{5}-2\left(\frac{2}{5}\right)}$ | $=\frac{\frac{2}{5}+1}{3\left(\frac{2}{5}\right)-6}-\frac{5\left(\frac{2}{5}\right)}{6}$ |
| $=\frac{1}{\frac{2}{5}-\frac{10}{5}}$ | $=\frac{\frac{2}{5}+\frac{5}{5}}{\frac{6}{5}-\frac{30}{5}}-\frac{2}{6}$ |
| $=\frac{1}{\frac{-8}{5}}$ | $=\frac{\frac{7}{5}}{\frac{-24}{5}}-\frac{1}{3}$ |
| $=-\frac{5}{8}$ | $=-\frac{7}{24}-\frac{8}{24}$ |
| | $=-\frac{15}{24}$ |
| | $=-\frac{5}{8}$ |

LS = RS

Therefore, the solution is $x=\frac{2}{5}$.

**69.**

$$\frac{2x+2}{x^2+2x-15}+\frac{2}{x-3}=\frac{1}{x+5}$$

$$\frac{2x+2}{(x+5)(x-3)}+\frac{2}{x-3}=\frac{1}{x+5}$$

$$(x+5)(x-3)\times\left[\frac{2x+2}{(x+5)(x-3)}+\frac{2}{x-3}\right]=(x+5)(x-3)\times\left[\frac{1}{x+5}\right]$$

$$2x+2+2(x+5)=1(x-3)$$

$$2x+2+2x+10=x-3$$

$$4x+12=x-3$$

$$3x=-15$$

$$x=-5$$

When $x=-5$, $\frac{1}{x+5}$ is undefined; therefore, there is no solution.

## Section 3: Activity 2 (continued)

**74.**

$$\frac{2}{s^2+5s+4}-1=\frac{1}{s+1}$$

$$\frac{2}{(s+1)(s+4)}-1=\frac{1}{s+1}$$

$$(s+1)(s+4)\times\left[\frac{2}{(s+1)(s+4)}-1\right]=(s+1)(s+4)\times\left(\frac{1}{s+1}\right)$$

$$2-1(s+1)(s+4)=1(s+4)$$

$$2-\left(s^2+5s+4\right)=s+4$$

$$2-s^2-5s-4=s+4$$

$$-s^2-5s-2=s+4$$

$$-s^2-6s-6=0$$

$$-1\left(-s^2-6s-6\right)=-1(0)$$

$$s^2+6s+6=0$$

$$\therefore\ s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-6\pm\sqrt{6^2-4(1)(6)}}{2(1)}$$

$$=\frac{-6\pm\sqrt{36-24}}{2}$$

$$=\frac{-6\pm\sqrt{12}}{2}$$

$$=\frac{-6\pm2\sqrt{3}}{2}$$

$$=-3\pm\sqrt{3}$$

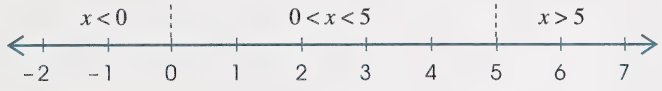
*Note:*
$a=1$
$b=6$
$c=6$

$$\therefore\ s=-3+\sqrt{3} \quad \text{or} \quad s=-3-\sqrt{3}$$
$$\doteq-1.3 \qquad\qquad \doteq-4.7$$

The solutions are approximately $s=-1.3$ and $s=-4.7$.

**77.** $\frac{x}{x-5} > 0$

The numerator is 0 when $x = 0$. The denominator is 0 when $x = 5$. So, the critical numbers are 0 and 5. There are three test intervals: $x < 0$, $0 < x < 5$, and $x > 5$.




In the test interval $x < 0$, use the point $x = -1$.

| LS | RS |
|---|---|
| $\frac{x}{x-5}$ | 0 |
| $= \frac{-1}{-1-5}$ | |
| $= \frac{1}{6}$ | |

LS > RS

The values in this interval satisfy the inequality.

In the test interval $0 < x < 5$, use the point $x = 2$.

| LS | RS |
|---|---|
| $\frac{x}{x-5}$ | 0 |
| $= \frac{2}{2-5}$ | |
| $= -\frac{2}{3}$ | |

LS $\not>$ RS

The values in this interval do not satisfy the inequality.

## Section 3: Activity 2 (continued)

In the test interval $x > 5$, use the point $x = 6$.

| LS | RS |
|---|---|
| $\frac{x}{x-5}$ | $0$ |
| $= \frac{6}{6-5}$ | |
| $= 6$ | |

LS $>$ RS

The values in this interval satisfy the inequality.

Check the two critical numbers, 0 and 5, which have not been tested yet.

For $x = 0$,

| LS | RS |
|---|---|
| $\frac{x}{x-5}$ | $0$ |
| $= \frac{0}{0-5}$ | |
| $= \frac{0}{-5}$ | |
| $= 0$ | |

LS $\not>$ RS

For $x = 5$,

| LS | RS |
|---|---|
| $\frac{x}{x-5}$ | $0$ |
| $= \frac{5}{5-5}$ | |
| $= \frac{5}{0}$ | |
| = undefined | |

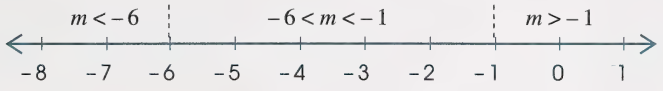
LS $\not>$ RS

Neither critical number is included in the solution.

Therefore, the solution is $x < 0$ or $x > 5$.

**82.** $\frac{m+6}{m+1} \leq 0$

The numerator is zero when $m = -6$. The denominator is zero when $m = -1$. So, the critical numbers are $-6$ and $-1$. There are three test intervals: $m < -6$, $-6 < m < -1$, and $m > -1$.




In the test interval $x < -6$, use the point $m = -7$.

| LS | RS |
|---|---|
| $\frac{m+6}{m+1}$ | 0 |
| $= \frac{-7+6}{-7+1}$ | |
| $= \frac{-1}{-6}$ | |
| $= \frac{1}{6}$ | |

LS $\not\leq$ RS

The values in this interval do not satisfy the inequality.

In the test interval $-6 < m < -1$, use the point $m = -5$.

| LS | RS |
|---|---|
| $\frac{m+6}{m+1}$ | 0 |
| $= \frac{-5+6}{-5+1}$ | |
| $= \frac{1}{-4}$ | |
| $= -\frac{1}{4}$ | |

LS $\leq$ RS

The values in this interval satisfy the inequality.

## Section 3: Activity 2 (continued)

In the test interval $m > -1$, use the point $m = 0$.

| LS | RS |
|---|---|
| $\frac{m+6}{m+1}$ | 0 |
| $= \frac{0+6}{0+1}$ | |
| $= 6$ | |

$$\text{LS} \not\leq \text{RS}$$

The values in this interval do not satisfy the inequality.

Check the critical numbers, $-6$ and $-1$, which have not been tested.

For $m = -6$,

| LS | RS |
|---|---|
| $\frac{m+6}{m+1}$ | 0 |
| $= \frac{-6+6}{-6+1}$ | |
| $= \frac{0}{-5}$ | |
| $= 0$ | |

$$\text{LS} \leq \text{RS}$$

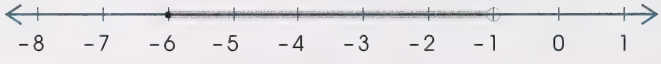
For $m = -1$,

| LS | RS |
|---|---|
| $\frac{m+6}{m+1}$ | 0 |
| $= \frac{-1+6}{-1+1}$ | |
| $= \frac{5}{0}$ | |
| = undefined | |

$$\text{LS} \not\leq \text{RS}$$

Only the critical number $-6$ satisfies the inequality.

Therefore, the solution is $-6 \leq m < -1$.



**95.** $\dfrac{4w}{w+1} > w+1$

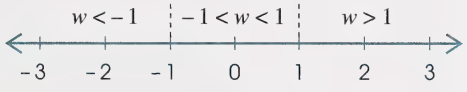
The inequality is not defined when $w=-1$; therefore, this is one of the points used to determine the test intervals.

To find the other points, solve the equation.

$$\begin{aligned}
\frac{4w}{w+1} &= w+1 \\
4w &= (w+1)(w+1) \\
4w &= w^2+2w+1 \\
0 &= w^2-2w+1 \\
w^2-2w+1 &= 0 \\
(w-1)^2 &= 0 \\
w-1 &= 0 \\
w &= 1
\end{aligned}$$

The points 1 and $-1$ are the critical numbers.

There are three test intervals: $w<-1$, $-1<w<1$, and $w>1$.




In the test interval $w<-1$, use the point $w=-2$.

| LS | RS |
|---|---|
| $\dfrac{4w}{w+1}$ | $w+1$ |
| $=\dfrac{4(-2)}{-2+1}$ | $=-2+1$ |
| $=\dfrac{-8}{-1}$ | $=-1$ |
| $=8$ | |

LS > RS

The values in this interval satisfy the inequality.

## Section 3: Activity 2 (continued)

In the test interval $-1 < w < 1$, use the point $w = 0$.

| LS | RS |
|---|---|
| $\frac{4w}{w+1}$ | $w+1$ |
| $= \frac{4(0)}{0+1}$ | $= 0+1$ |
| $= 0$ | $= 1$ |

LS $\not>$ RS

The values in this interval do not satisfy the inequality.

In the test interval $w > 1$, use the point $w = 2$.

| LS | RS |
|---|---|
| $\frac{4w}{w+1}$ | $w+1$ |
| $= \frac{4(2)}{2+1}$ | $= 2+1$ |
| $= \frac{8}{3}$ | $= 3$ |
| $= 2\frac{2}{3}$ | |

LS $\not>$ RS

The values in this interval do not satisfy the inequality.

Check the critical numbers, $\pm 1$, which have not been tested.

For $w = -1$,

| LS | RS |
|---|---|
| $\frac{4w}{w+1}$ | $w+1$ |
| $= \frac{4(-1)}{-1+1}$ | $= -1+1$ |
| $= \frac{-4}{0}$ | $= 0$ |
| = undefined | |

LS $\not>$ RS

For $w = 1$,

| LS | RS |
|---|---|
| $\frac{4w}{w+1}$ | $w+1$ |
| $= \frac{4(1)}{1+1}$ | $= 1+1$ |
| $= \frac{4}{2}$ | $= 2$ |
| $= 2$ | |

LS $\not>$ RS

Neither critical number is part of the solution.

Therefore, the solution is $w < -1$.

**b. Textbook questions 110, 111, 113, and 127 of "Applications and Problem Solving," pp. 311 and 312**

**110.** The number represented by $x$ is the smallest of the three numbers.

$$\therefore \quad \frac{1}{x} = \frac{1}{x+1} + \frac{1}{x+2}$$

$$x(x+1)(x+2) \times \frac{1}{x} = x(x+1)(x+2) \times \left[\frac{1}{x+1} + \frac{1}{x+2}\right]$$

$$(x+1)(x+2) = x(x+2) + x(x+1)$$

$$x^2 + 3x + 2 = x^2 + 2x + x^2 + x$$

$$x^2 + 3x + 2 = 2x^2 + 3x$$

$$2 = x^2$$

$$x = \pm\sqrt{2}$$

The values of $x$ are $\pm\sqrt{2}$.

## Section 3: Activity 2 (continued)

**111. a.** Use the relationship distance = speed × time, and set up a table.

| Driver | Distance (km) | Speed (km/h) | Time (h) |
|---|---|---|---|
| Laura | 300 | $s$ | $\frac{300}{s}$ |
| Mariko | 320 | $s+5$ | $\frac{320}{s+5}$ |

Because the times are equal,

$$\frac{300}{s} = \frac{320}{s+5}$$
$$300(s+5) = 320s$$
$$300s + 1500 = 320s$$
$$1500 = 20s$$
$$20s = 1500$$
$$s = 75$$

Laura's speed was 75 km/h and Mariko's speed was 80 km/h.

**b.** The driving time was $\frac{300 \text{ km}}{75 \text{ km/h}} = 4 \text{ h}$. Therefore, they both left home at 8:00 A.M.

**113.** Calculate the original mass of cashews in the tin.

$$40\% \text{ of } 250 = 0.40 \times 250$$
$$= 100$$

There are 100 g of cashews in the original mixture.

Let $m$ be the additional mass of cashews.

Therefore, the total mass of cashews is $100 + m$, and the total mass of the mixture is $250 + m$.

$\therefore$ Mass of cashews = 60% of contents of bowl

$$100 + m = 0.60(250 + m)$$
$$100 + m = 150 + 0.60m$$
$$0.40m = 50$$
$$m = 125$$

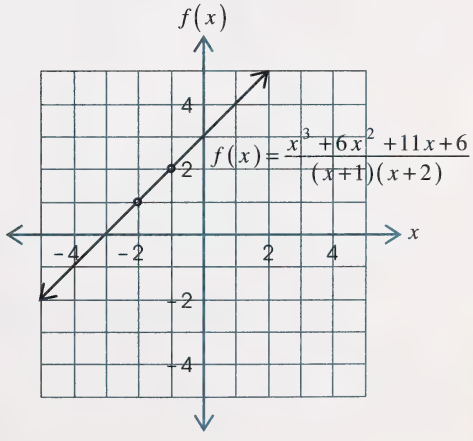
Therefore, 125 g of cashews must be added to the original mixture.

**127. a.** $f(x) = \dfrac{x^3 + 6x^2 + 11x + 6}{(x+1)(x+2)}$

$$= \frac{(x+1)(x+2)(x+3)}{(x+1)(x+2)}$$
$$= x + 3, \text{ where } x \neq -1, -2$$

Use a table of values to graph $f(x)$:

| $x$ | $f(x)$ |
|---|---|
| −3 | 0 |
| −2 | undefined |
| −1 | undefined |
| 0 | 3 |
| 1 | 4 |
| 2 | 5 |



## Section 3: Activity 2 (continued)

**b.** The graph of $g(x) = x + 3$ is continuous. It includes $(-2, 1)$ and $(-1, 2)$, which are not included in the graph of $f(x) = \frac{x^3 + 6x^2 + 11x + 6}{(x+1)(x+2)}$.

**4. Textbook question 126 of "Applications and Problem Solving," p. 312**

**126. a.** $v = 343$ m/s and $f_s = 310$ Hz,

$$\begin{aligned} f &= \frac{f_s}{1 \pm \frac{u}{v}} \\ &= \frac{310}{1 \pm \frac{u}{343}} \\ &= \frac{310}{\frac{343}{343} \pm \frac{u}{343}} \\ &= \frac{310}{\frac{343 \pm u}{343}} \\ &= \frac{310(343)}{343 \pm u} \\ &= \frac{106\,330}{343 \pm u} \end{aligned}$$

**b.** The $\pm$ sign appears in the function because the source could be moving towards you $(-)$ or away from you $(+)$.

**c.** If the firetruck is approaching the corner at 14 m/s, then

$$\begin{aligned} f &= \frac{106\,330}{343 - u} \\ &= \frac{106\,330}{343 - 14} \\ &\doteq 323 \end{aligned}$$

The siren frequency of the approaching firetruck is approximately 323 Hz.

If the firetruck is leaving the corner at 14 m/s, then

$$\begin{aligned} f &= \frac{106\,330}{343+u} \\ &= \frac{106\,330}{343+14} \\ &\doteq 298 \end{aligned}$$

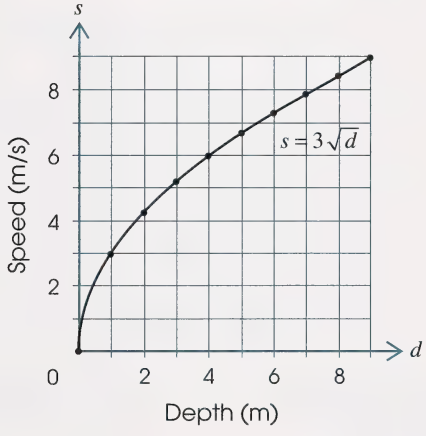
The siren frequency of the truck as it leaves the corner is approximately 298 Hz.

The pitch of the siren is higher as the truck approaches and lower as the truck moves away.

## Section 3: Activity 3

**1. a. Textbook question, "Explore: Draw a Graph," p. 314**

| *d* | *s* |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | $3\sqrt{2} \doteq 4.2$ |
| 3 | $3\sqrt{3} \doteq 5.2$ |
| 4 | 6 |
| 5 | $3\sqrt{5} \doteq 6.7$ |
| 6 | $3\sqrt{6} \doteq 7.3$ |
| 7 | $3\sqrt{7} \doteq 7.9$ |
| 8 | $6\sqrt{2} \doteq 8.5$ |
| 9 | 9 |




**b. Textbook questions 1 to 4 of "Inquire," p. 314**

**1.** The domain is $d \geq 0$ because depth is non-negative.

**2.** The range is $s \geq 0$ because speed is non-negative.

## Section 3: Activity 3 (continued)

**3. a.** $s = 3\sqrt{d}$

$= 3\sqrt{100}$

$= 3(10)$

$= 30$

In 100-m deep water, the speed of a tsunami wave is 30 m/s.

**b.** $30 \text{ m/s} = \frac{30 \cancel{\text{m}}}{\cancel{\text{s}}} \times \frac{1 \text{ km}}{1000 \cancel{\text{m}}} \times \frac{3600 \cancel{\text{s}}}{1 \text{ h}}$

$= \frac{108\,000 \text{ km}}{1000 \text{ h}}$

$= 108 \text{ km/h}$

The speed of the tsunami wave is 108 km/h.

**4. a.** $s = 3\sqrt{d}$

$= 3\sqrt{4600}$

$= 30\sqrt{46}$

$\doteq 203$

The speed of a tsunami wave in the Pacific Basin is about 203 m/s.

**b.** $30\sqrt{46} \text{ m/s} = \frac{30\sqrt{46} \cancel{\text{m}}}{\cancel{\text{s}}} \times \frac{1 \text{ km}}{1000 \cancel{\text{m}}} \times \frac{3600 \cancel{\text{s}}}{1 \text{ h}}$

$= \frac{108\,000\sqrt{46} \text{ km}}{1000 \text{ h}}$

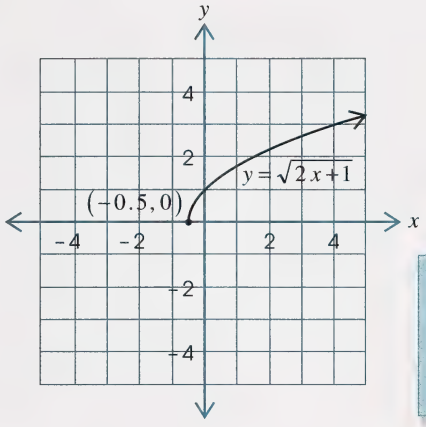
$\doteq 732 \text{ km/h}$

The speed of the tsunami wave is about 732 km/h.

**2. Textbook questions 3, 11, and 19 of "Practice," p. 323**

**3.**

| $x$ | $y$ |
|---|---|
| $-0.5$ | 0 |
| 0 | 1 |
| 1 | $\sqrt{3} \doteq 1.7$ |
| 2 | $\sqrt{5} \doteq 2.2$ |
| 3 | $\sqrt{7} \doteq 2.6$ |
| 4 | 3 |
| 5 | $\sqrt{11} \doteq 3.3$ |




$\sqrt{2x+1}$ *is defined when*

$2x+1 \geq 0$

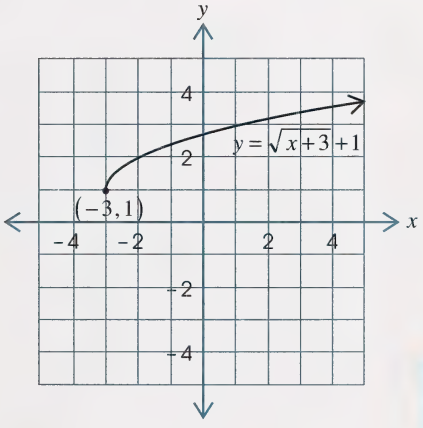
$2x \geq -1$

$x \geq -0.5$

The domain is $x \geq -0.5$, and the range is $y \geq 0$.

**11.**

| $x$ | $y$ |
|---|---|
| $-3$ | 1 |
| $-2$ | 2 |
| $-1$ | $\sqrt{2}+1 \doteq 2.4$ |
| 0 | $\sqrt{3}+1 \doteq 2.7$ |
| 1 | 3 |
| 2 | $\sqrt{5}+1 \doteq 3.2$ |
| 3 | $\sqrt{6}+1 \doteq 3.4$ |
| 4 | $\sqrt{7}+1 \doteq 3.6$ |




$\sqrt{x+3}$ *is defined when*

$x+3 \geq 0$

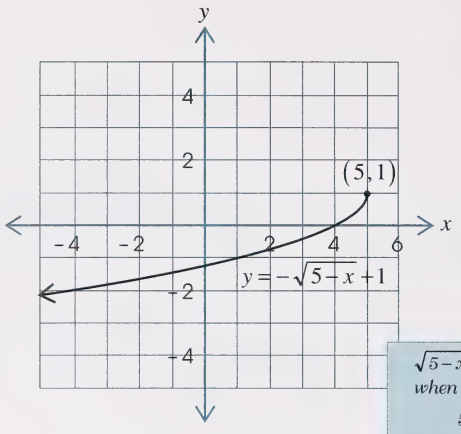
$x \geq -3$

The domain is $x \geq -3$, and the range is $y \geq 1$.

## Section 3: Activity 3 (continued)

**19.**

| $x$ | $y$ |
|---|---|
| 5 | 1 |
| 4 | 0 |
| 3 | $-\sqrt{2}+1 \doteq -0.4$ |
| 2 | $-\sqrt{3}+1 \doteq -0.7$ |
| 1 | $-1$ |
| 0 | $-\sqrt{5}+1 \doteq -1.2$ |
| $-1$ | $-\sqrt{6}+1 \doteq -1.4$ |
| $-2$ | $-\sqrt{7}+1 \doteq -1.6$ |




*$\sqrt{5-x}$ is defined when*

$5-x \geq 0$

$5 \geq x$

$x \leq 5$

The domain is $x \leq 5$, and the range is $y \leq 1$.

**3. a. Textbook questions 30, 31, 44, 48, 53, and 60 of "Practice," p. 323**

**30.** $\sqrt{4-3x}-2=0$

$\sqrt{4-3x}=2$

$\left(\sqrt{4-3x}\right)^2=2^2$

$4-3x=4$

$-3x=0$

$x=0$

**Check**

| LS | RS |
|---|---|
| $\sqrt{4-3x}-2$ | 0 |
| $=\sqrt{4-3(0)}-2$ | |
| $=\sqrt{4}-2$ | |
| $=2-2$ | |
| $=0$ | |

LS = RS

*Note: Use the principal square root only in a check.*

The solution is $x=0$.

**31.** $\sqrt{x-3}+6=2$

$$\sqrt{x-3}=-4$$
$$\left(\sqrt{x-3}\right)^2=(-4)^2$$
$$x-3=16$$
$$x=19$$

Because $x=19$ is an extraneous root, there is no solution.

**Check**

| LS | RS |
|---|---|
| $\sqrt{x-3}+6$ | 2 |
| $=\sqrt{19-3}+6$ | |
| $=\sqrt{16}+6$ | |
| $=4+6$ | |
| $=10$ | |

LS $\neq$ RS

**44.** $\sqrt{y+3}-3\sqrt{y-1}=0$

$$\sqrt{y+3}=3\sqrt{y-1}$$ ← Separate the radicals.
$$\left(\sqrt{y+3}\right)^2=\left(3\sqrt{y-1}\right)^2$$
$$y+3=9(y-1)$$
$$y+3=9y-9$$
$$-8y=-12$$
$$y=\frac{-12}{-8}$$
$$=\frac{3}{2}$$

The solution is $y=\frac{3}{2}$.

**Check**

| LS | RS |
|---|---|
| $\sqrt{y+3}-3\sqrt{y-1}$ | 0 |
| $=\sqrt{\frac{3}{2}+3}-3\sqrt{\frac{3}{2}-1}$ | |
| $=\sqrt{\frac{9}{2}}-3\sqrt{\frac{1}{2}}$ | |
| $=\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}$ | |
| $=0$ | |

LS = RS

**48.** $\sqrt{x}-\sqrt{x-5}=5$

$$\sqrt{x}=5+\sqrt{x-5}$$
$$\left(\sqrt{x}\right)^2=\left(5+\sqrt{x-5}\right)^2$$
$$x=25+10\sqrt{x-5}+x-5$$
$$x=x+20+10\sqrt{x-5}$$
$$-20=10\sqrt{x-5}$$
$$-2=\sqrt{x-5}$$
$$(-2)^2=\left(\sqrt{x-5}\right)^2$$
$$4=x-5$$
$$x-5=4$$
$$x=9$$

Because $x=9$ is an extraneous root, there is no solution.

**Check**

| LS | RS |
|---|---|
| $\sqrt{x}-\sqrt{x-5}$ | 5 |
| $=\sqrt{9}-\sqrt{9-5}$ | |
| $=3-\sqrt{4}$ | |
| $=3-2$ | |
| $=1$ | |

LS $\neq$ RS

## Section 3: Activity 3 (continued)

**53.**

$$3 = \sqrt{x+3} + \sqrt{x}$$
$$3 - \sqrt{x} = \sqrt{x+3}$$
$$\left(3 - \sqrt{x}\right)^2 = \left(\sqrt{x+3}\right)^2$$
$$9 - 6\sqrt{x} + x = x + 3$$
$$-6\sqrt{x} = -6$$
$$\sqrt{x} = 1$$
$$\left(\sqrt{x}\right)^2 = 1^2$$
$$x = 1$$

The solution is $x = 1$.

**Check**

| LS | RS |
|---|---|
| 3 | $\sqrt{x+3} + \sqrt{x}$ |
| | $= \sqrt{1+3} + \sqrt{1}$ |
| | $= \sqrt{4} + 1$ |
| | $= 2 + 1$ |
| | $= 3$ |

LS = RS

**60.** $\sqrt{2x+1} - x = 0$

$$\sqrt{2x+1} = x$$
$$\left(\sqrt{2x+1}\right)^2 = x^2$$
$$2x + 1 = x^2$$
$$0 = x^2 - 2x - 1$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4+4}}{2}$$
$$= \frac{2 \pm \sqrt{8}}{2}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$
$$= 1 \pm \sqrt{2}$$

*Note:*
$a = 1$
$b = -2$
$c = -1$

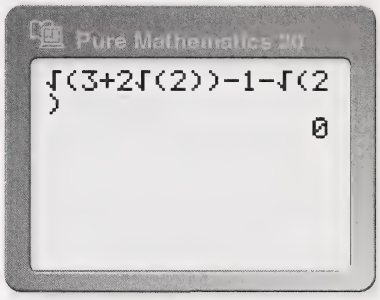
$\therefore x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$
$\doteq 2.4$ $\doteq -0.4$

**Check**

For $x = 1 + \sqrt{2}$,

| LS | RS |
|---|---|
| $\sqrt{2x+1} - x$ | 0 |
| $= \sqrt{2\left(1+\sqrt{2}\right)+1} - \left(1+\sqrt{2}\right)$ | |
| $= \sqrt{2+2\sqrt{2}+1} - 1 - \sqrt{2}$ | |
| $= \sqrt{3+2\sqrt{2}} - 1 - \sqrt{2}$ | |
| $= 0$ | |

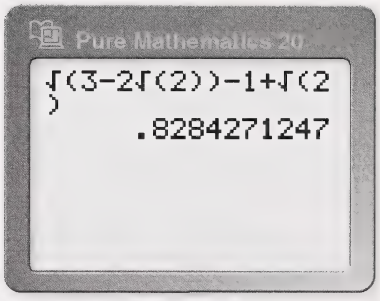
LS = RS




For $x = 1 - \sqrt{2}$,

| LS | RS |
|---|---|
| $\sqrt{2x+1} - x$ | 0 |
| $= \sqrt{2\left(1-\sqrt{2}\right)+1} - \left(1-\sqrt{2}\right)$ | |
| $= \sqrt{2-2\sqrt{2}+1} - 1 + \sqrt{2}$ | |
| $= \sqrt{3-2\sqrt{2}} - 1 + \sqrt{2}$ | |
| $= 0.828\,427\,124\,7\ldots$ | |

LS ≠ RS




The solution is $x = 1 + \sqrt{2}$, or approximately 2.4.

## Section 3: Activity 3 (continued)

**b. Textbook questions 109.a., 114, 119, 125, 129, and 140.a. of "Applications and Problem Solving," pp. 324 to 327**

**109. a.**

$$\sqrt{x+2} = \sqrt{x} + \sqrt{2}$$

$$\left(\sqrt{x+2}\right)^2 = \left(\sqrt{x} + \sqrt{2}\right)^2$$

$$x+2 = x + 2\sqrt{2x} + 2$$

$$0 = 2\sqrt{2x}$$

$$0 = \sqrt{2x}$$

$$0^2 = \left(\sqrt{2x}\right)^2$$

$$0 = 2x$$

$$2x = 0$$

$$x = 0$$

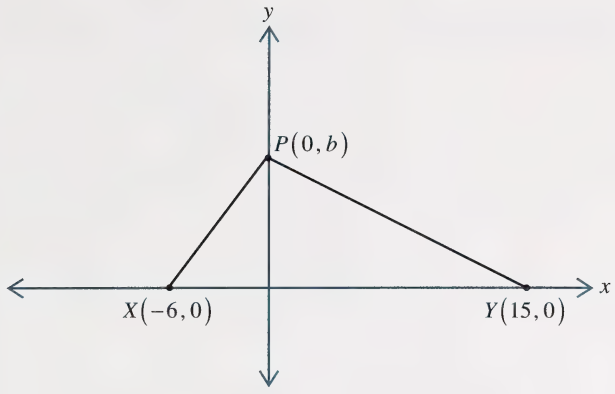
**Check**

| LS | RS |
|---|---|
| $\sqrt{x+2}$ | $\sqrt{x} + \sqrt{2}$ |
| $= \sqrt{0+2}$ | $= \sqrt{0} + \sqrt{2}$ |
| $= \sqrt{2}$ | $= \sqrt{2}$ |

LS = RS

The solution is $x = 0$.

**114.**




Let point $P$ be $(0, b)$, and use the distance formula to find the unknown.

$$PY - PX = 7$$

$$\sqrt{(15-0)^2+(0-b)^2} - \sqrt{(-6-0)^2+(0-b)^2} = 7$$

$$\sqrt{15^2+b^2} - \sqrt{(-6)^2+b^2} = 7$$

$$\sqrt{225+b^2} - \sqrt{36+b^2} = 7$$

$$\sqrt{225+b^2} = 7+\sqrt{36+b^2}$$

$$\left(\sqrt{225+b^2}\right)^2 = \left(7+\sqrt{36+b^2}\right)^2$$

$$225+b^2 = 49+14\sqrt{36+b^2}+36+b^2$$

$$b^2+225 = b^2+85+14\sqrt{36+b^2}$$

$$140 = 14\sqrt{36+b^2}$$

$$10 = \sqrt{36+b^2}$$

$$10^2 = \left(\sqrt{36+b^2}\right)^2$$

$$100 = 36+b^2$$

$$64 = b^2$$

$$b = \pm 8$$

The coordinates of point $P$ are $(0, \pm 8)$.

## Section 3: Activity 3 (continued)

**119. a.** $P = 0.2\sqrt{\ell}$

$= 0.2\sqrt{25}$

$= 0.2(5)$

$= 1.0$

The period of a pendulum 25 cm in length is 1 s.

**b.** $P = 0.2\sqrt{\ell}$

$4 = 0.2\sqrt{\ell}$

$\frac{4}{0.2} = \frac{0.2\sqrt{\ell}}{0.2}$

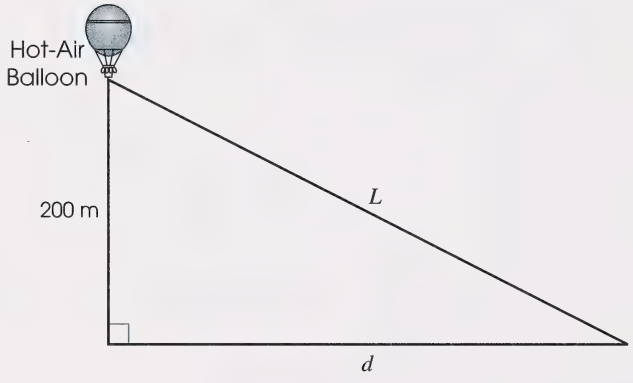
$20 = \sqrt{\ell}$

$20^2 = \left(\sqrt{\ell}\right)^2$

$400 = \ell$

The length of a pendulum with a period of 4 s is 400 cm or 4 m.

**125.**




**a.** $L^2 = d^2 + 200^2$

$L^2 = d^2 + 40\,000$

$d^2 + 40\,000 = L^2$

$d^2 = L^2 - 40\,000$

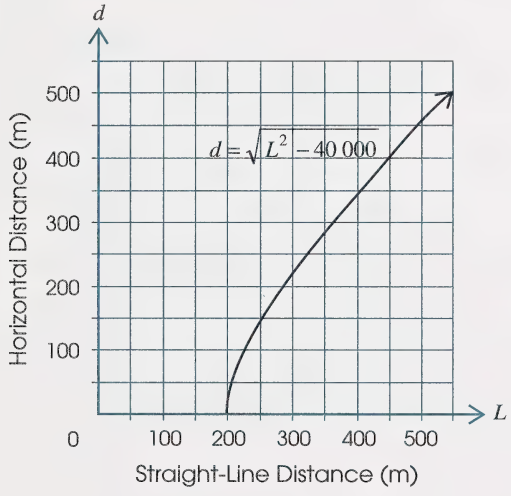
$d = \sqrt{L^2 - 40\,000}$

**b.** The function is defined when

$$L^2 - 40\,000 \geq 0$$
$$L^2 \geq 40\,000$$
$$\sqrt{L^2} \geq \sqrt{40\,000}$$
$$L \geq 200$$

The domain is $L \geq 200$ m.

**c.**




**d.** $d = \sqrt{L^2 - 40\,000}$

$= \sqrt{250^2 - 40\,000}$

$= \sqrt{62\,500 - 40\,000}$

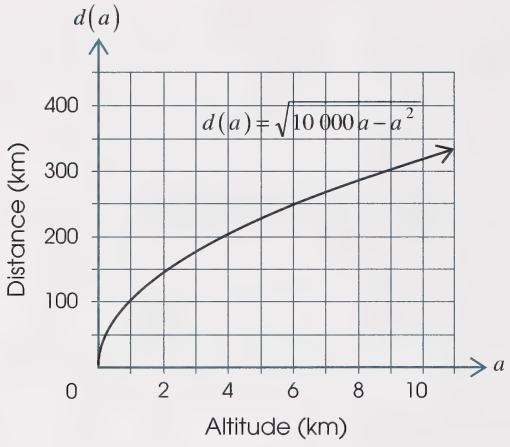
$= \sqrt{22\,500}$

$= 150$

When the straight-line distance from the observer to the balloon is 250 m, the horizontal distance from the observer to the balloon is 150 m.

## Section 3: Activity 3 (continued)

**129. a.**




**b.** Answers may vary. A sample answer is given.

The domain is $0 \le a \le 20$, and the range is $0 \le d(a) \le 450$.

**c.** $d(a)=\sqrt{10\,000a-a^2}$

$d(10)=\sqrt{10\,000(10)-10^2}$

$=\sqrt{100\,000-100}$

$=\sqrt{99\,900}$

$\doteq 316$

10 000 m = 10 km

At an altitude of 10 000 m, you can see approximately 316 km.

**d.** Answers may vary. A sample answer is given.

Suppose your school is in Lloydminster. The pilot of an aircraft, flying at an altitude of 10 000 m above your school could see Edmonton, Saskatoon, Fort McMurray, and Rosetown.

**140. a.** $\sqrt[3]{x+3}+2=4$

$$\sqrt[3]{x+3}=2$$
$$\left(\sqrt[3]{x+3}\right)^3=2^3$$
$$x+3=8$$
$$x=5$$

The solution is $x=5$.

**Check**

| LS | RS |
|---|---|
| $\sqrt[3]{x+3}+2$ | 4 |
| $=\sqrt[3]{5+3}+2$ | |
| $=\sqrt[3]{8}+2$ | |
| $=2+2$ | |
| $=4$ | |

LS = RS

**4. a. Textbook questions 68, 73, 78, 83, and 95 of "Practice," pp. 323 and 324**

**68.** $\sqrt{x+1}<3$

$$\left(\sqrt{x+1}\right)^2<3^2$$
$$x+1<9$$
$$x<8$$

Check that 8 is a solution of the corresponding equation, $\sqrt{x+1}=3$.

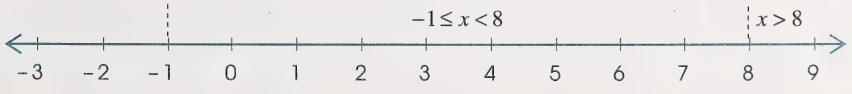
| LS | RS |
|---|---|
| $\sqrt{x+1}$ | 3 |
| $=\sqrt{8+1}$ | |
| $=\sqrt{9}$ | |
| $=3$ | |

LS = RS

So, 8 is a real solution of the corresponding equation and is not an extraneous solution. It can be used to determine the test intervals. Also, because the radicand in $\sqrt{x+1}$ cannot be negative, the restriction on the variable in the original inequality is

$$x+1\geq 0$$
$$x\geq -1$$

Therefore, there are two test intervals: $-1\leq x<8$ and $x>8$.



## Section 3: Activity 3 (continued)

In the test interval $-1 \le x < 8$, use the point $x = 0$.

| LS | RS |
|---|---|
| $\sqrt{x+1}$ | 3 |
| $=\sqrt{0+1}$ | |
| $=1$ | |

$$\text{LS} < \text{RS}$$

The original inequality is satisfied in this interval.

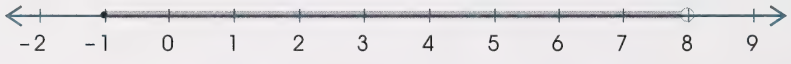
In the test interval $x > 8$, use the point $x = 15$.

| LS | RS |
|---|---|
| $\sqrt{x+1}$ | 3 |
| $=\sqrt{15+1}$ | |
| $=\sqrt{16}$ | |
| $=4$ | |

$$\text{LS} \not< \text{RS}$$

The original inequality is not satisfied in this interval.

Recall that $x = 8$ satisfies $\sqrt{x+1} = 3$ and not $\sqrt{x+1} < 3$. Therefore, the solution is $-1 \le x < 8$.




**73.** $\sqrt{x+2} - \sqrt{4-x} > 0$

$$\sqrt{x+2} > \sqrt{4-x}$$

$$\left(\sqrt{x+2}\right)^2 > \left(\sqrt{4-x}\right)^2$$

$$x+2 > 4-x$$

$$2x > 2$$

$$x > 1$$

Check that 1 is a root of the corresponding equation, $\sqrt{x+2} - \sqrt{4-x} = 0$.

| LS | RS |
|---|---|
| $\sqrt{x+2}-\sqrt{4-x}$ | 0 |
| $=\sqrt{1+2}-\sqrt{4-1}$ | |
| $=\sqrt{3}-\sqrt{3}$ | |
| $=0$ | |

LS = RS

So, 1 is a solution of the corresponding equation. It can be used to determine the test intervals. Also, because radicands cannot be negative, the restrictions on the variable in the original inequality are

$x+2\geq 0$ and $4-x\geq 0$

$x\geq -2$ $x\leq 4$

Therefore, there are two intervals: $-2\leq x<1$ and $1<x\leq 4$.

In the test interval $-2\leq x<1$, use the point $x=0$.

| LS | RS |
|---|---|
| $\sqrt{x+2}-\sqrt{4-x}$ | 0 |
| $=\sqrt{0+2}-\sqrt{4-0}$ | |
| $=\sqrt{2}-\sqrt{4}$ | |
| $\doteq -0.6$ | |

LS $\not>$ RS

The original inequality is not satisfied in this interval.

## Section 3: Activity 3 (continued)

In the test interval $1 < x \leq 4$, use the point $x = 2$.

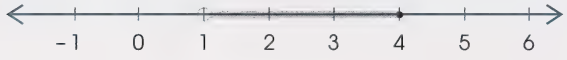
| LS | RS |
|---|---|
| $\sqrt{x+2} - \sqrt{4-x}$ | 0 |
| $= \sqrt{2+2} - \sqrt{4-2}$ | |
| $= \sqrt{4} - \sqrt{2}$ | |
| $\doteq 0.6$ | |

LS > RS

The original inequality is satisfied in this interval.

Recall that $x = 1$ satisfies the equation $\sqrt{x+2} - \sqrt{4-x} = 0$ and not the inequality.

Therefore, the solution is $1 < x \leq 4$.




**78.** $\sqrt{1-x} + 3 \geq x$

$$\sqrt{1-x} \geq x - 3$$

$$\left(\sqrt{1-x}\right)^2 \geq (x-3)^2$$

$$1 - x \geq x^2 - 6x + 9$$

$$0 \geq x^2 - 5x + 8$$

$$x^2 - 5x + 8 \leq 0$$

The corresponding radical equation is $x^2 - 5x + 8 = 0$. If it has real roots, they are obtained using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 32}}{2}$$

$$= \frac{5 \pm \sqrt{-7}}{2}$$

*Note:*

$a = 1$

$b = -5$

$c = 8$

Therefore, there are no real roots. The expression $x^2 - 5x + 8 \neq 0$. It will be always positive or always negative. Use $x = 0$ to test the sign of the expression.

$0^2 - 5(0) + 8 = 8$ ← positive

Therefore, $x^2 - 5x + 8$ is positive for all real numbers; it cannot be used to determine test intervals.

Instead, the test intervals are determined from restrictions on the variable in the original inequality. Since radicands cannot be negative,

$$1 - x \geq 0$$
$$1 \geq x$$
$$x \leq 1$$

Therefore, the test interval is $x \leq 1$.

In the test interval $x \leq 1$, use the point $x = 0$.

| LS | RS |
|---|---|
| $\sqrt{1-x}+3$ | $x$ |
| $=\sqrt{1-0}+3$ | $=0$ |
| $=1+3$ | |
| $=4$ | |

LS ≥ RS

The original inequality is satisfied in this interval.

The solution is $x \leq 1$.

## Section 3: Activity 3 (continued)

**83.** $\sqrt{2x-1} < 3x-1$

$$\left(\sqrt{2x-1}\right)^2 < (3x-1)^2$$

$$2x-1 < 9x^2-6x+1$$

$$0 < 9x^2-8x+2$$

The corresponding equation is $9x^2-8x+2=0$. If it has real roots, they are obtained using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-8) \pm \sqrt{(-8)^2-4(9)(2)}}{2(9)}$$

$$= \frac{8 \pm \sqrt{64-72}}{18}$$

$$= \frac{8 \pm \sqrt{-8}}{18}$$

*Note:*
$a=9$
$b=-8$
$c=2$

Therefore, the equation has no real roots. The expression $9x^2-8x+2$ will either be always positive or always negative. Use the point $x=0$ to test the sign of the equation.

$9(0)^2-8(0)+2=2$ ← positive

The expression $9x^2-8x+2$ is always positive; it cannot be used to determine test intervals.

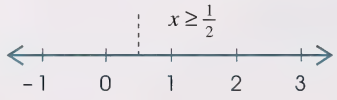
Test intervals are obtained from the radicand in the original inequality.

$$2x-1 \geq 0$$

$$2x \geq 1$$

$$x \geq \frac{1}{2}$$

Therefore, the test interval is $x \geq \frac{1}{2}$.



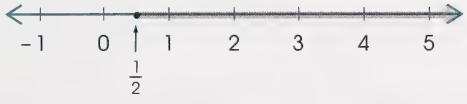
In the test interval $x \geq \frac{1}{2}$, use the point $x = 1$.

| LS | RS |
|---|---|
| $\sqrt{2x-1}$ | $3x-1$ |
| $=\sqrt{2(1)-1}$ | $=3(1)-1$ |
| $=\sqrt{1}$ | $=3-1$ |
| $=1$ | $=2$ |

$$\text{LS} < \text{RS}$$

The original inequality is satisfied in this interval.

The solution is $x \geq \frac{1}{2}$.




**95.**

$$\sqrt{x+2} > \frac{x}{x+2}$$

$$\left(\sqrt{x+2}\right)^2 > \left(\frac{x}{x+2}\right)^2 \quad \leftarrow \text{Square both sides.}$$

$$x+2 > \frac{x^2}{(x+2)^2}$$

$$(x+2)(x+2)^2 > x^2 \quad \leftarrow \text{Multiply both sides by } (x+2)^2.$$

$$(x+2)(x^2+4x+4) > x^2$$

$$x^3+4x^2+4x+2x^2+8x+8 > x^2$$

$$x^3+6x^2+12x+8 > x^2$$

$$x^3+5x^2+12x+8 > 0$$

*Recall that multiplying an inequality by a positive value does not change an inequality.*

Use the Integral Zero Theorem to find a factor of $x^3+5x^2+12x+8$.

$$\begin{aligned} f(-1) &= (-1)^3+5(-1)^2+12(-1)+8 \\ &= -1+5-12+8 \\ &= 0 \end{aligned}$$

## Section 3: Activity 3 (continued)

So, $x+1$ is a factor.

Use synthetic division to find another factor.

$$\begin{array}{r|cccc} -1 & 1 & 5 & 12 & 8 \\ & & -1 & -4 & -8 \\ \hline & 1 & 4 & 8 & 0 \end{array}$$

Another factor is $x^2+4x+8$, which cannot be factored further.

So, $x^3+5x^2+12x+8=(x+1)\left(x^2+4x+8\right)$.

Check that $x=-1$ is a root of $\sqrt{x+2}=\frac{x}{x+2}$.

| LS | RS |
|---|---|
| $\sqrt{x+2}$ | $\frac{x}{x+2}$ |
| $=\sqrt{-1+2}$ | $=\frac{-1}{-1+2}$ |
| $=\sqrt{1}$ | $=\frac{-1}{1}$ |
| $=1$ | $=-1$ |

$$\text{LS} \neq \text{RS}$$

So, $x=-1$ is an extraneous root. It cannot be used to decide on test intervals.

Now, $\sqrt{x+2}$ is defined for $x+2\geq 0$, or $x\geq -2$, and $\frac{x}{x+2}$ is not defined at $x=-2$.

So, the test interval is $x>-2$.

In the test interval $x > -2$, use the point $x = 7$.

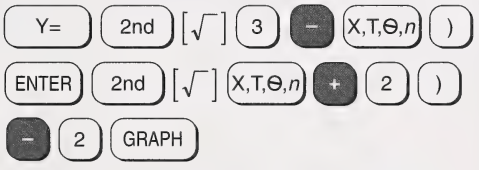
| LS | RS |
|---|---|
| $\sqrt{x+2}$ | $\frac{x}{x+2}$ |
| $= \sqrt{7+2}$ | $= \frac{7}{7+2}$ |
| $= \sqrt{9}$ | |
| $= 3$ | $= \frac{7}{9}$ |

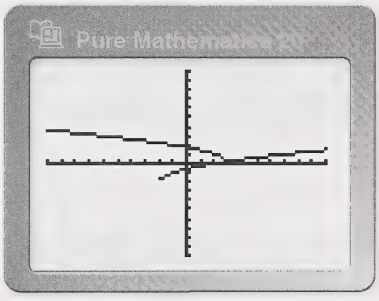
LS > RS

The original inequality is satisfied in this interval. Therefore, the solution is $x > -2$.

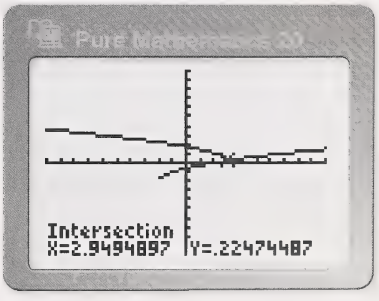
**b. Textbook questions 103.a., 122, 132.a., and 132.c. of "Applications and Problem Solving," pp. 324 to 326**

**103. a.** Using the standard window settings, graph $y = \sqrt{3-x}$ and $y = \sqrt{x+2} - 2$ on your graphing calculator.




Use the Intersect feature from the CALCULATE menu to determine where the curves cross.




The graph $y = \sqrt{3-x}$ lies above the graph $y = \sqrt{x+2} - 2$ to the left of $x \doteq 2.9$. Also, the values of $x < -2$ are not permissible because $\sqrt{x+2}$ would be undefined.

Therefore, the solution is approximately $-2 \leq x < 2.9$.

## Section 3: Activity 3 (continued)

**122.** Let $n$ be the number.

$$\therefore \sqrt{n+2} \leq 10$$
$$\left(\sqrt{n+2}\right)^2 \leq 10^2$$
$$n+2 \leq 100$$
$$n \leq 98$$

*"At most" means "less than or equal to."*

Check that $n = 98$ satisfies $\sqrt{n+2} = 10$.

| LS | RS |
|---|---|
| $\sqrt{n+2}$ | 10 |
| $= \sqrt{98+2}$ | |
| $= \sqrt{100}$ | |
| $= 10$ | |

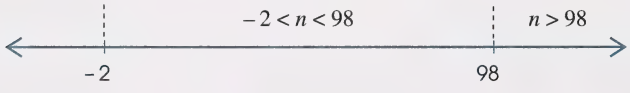
LS = RS

So, $n = 98$ can be used to determine test intervals.

Also, $\sqrt{n+2}$ is defined when

$$n+2 \geq 0$$
$$n \geq -2$$

Therefore, there are two test intervals: $-2 \leq n < 98$ and $n > 98$.




In the test interval $-2 \leq n < 98$, use the point $n = 0$.

| LS | RS |
|---|---|
| $\sqrt{n+2}$ | 10 |
| $= \sqrt{0+2}$ | |
| $= \sqrt{2}$ | |

LS ≤ RS

The original inequality is satisfied in this interval.

In the test interval $n \geq 98$, use the point $n = 100$.

| LS | RS |
|---|---|
| $\sqrt{n+2}$ | 10 |
| $= \sqrt{100+2}$ | |
| $= \sqrt{102}$ | |

$$\text{LS} \not\leq \text{RS}$$

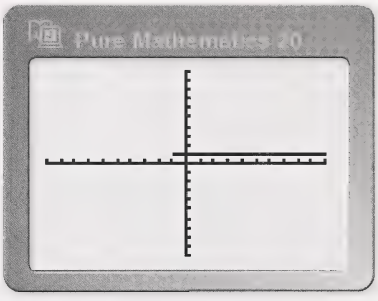
The original inequality is not satisfied in this interval.

Recall that $n = 98$ satisfies the equation and the inequality; therefore, the values of $n$, for which the square root of 2 more than $n$ is at most 10, are $-2 \leq n \leq 98$.

**132. a.** To solve $\sqrt{x+1} \geq 0$ directly, make sure your graphing calculator is in Dot mode.

Y= 2nd [√] X,T,Θ,n + 1 )

2nd [ TEST ] 4 0 GRAPH




Because the inequality is $\geq$, include $-1$ in your solution.

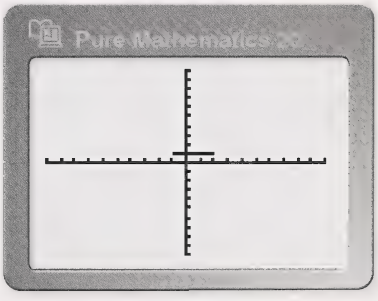
The solution is $x \geq -1$.

**c.** To solve $2x - \sqrt{3x+3} < 1$ directly, make sure your graphing calculator is in Dot mode.

Y= 2 X,T,Θ,n − 2nd [√] 3

X,T,Θ,n + 3 ) 2nd [ TEST ] 5

1 GRAPH



## Section 3: Activity 3 (continued)

The endpoints are $x = -1$ and $x = 2$. Check to see if either satisfies the original inequality.

For $x = -1$,

| LS | RS |
|---|---|
| $2x - \sqrt{3x+3}$ | 1 |
| $= 2(-1) - \sqrt{3(-1)+3}$ | |
| $= -2 - \sqrt{0}$ | |
| $= -2$ | |

$$\text{LS} < \text{RS}$$

So, $x = -1$ satisfies the original inequality.

For $x = 2$,

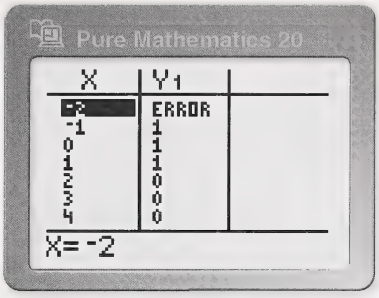
| LS | RS |
|---|---|
| $2x - \sqrt{3x+3}$ | 1 |
| $= 2(2) - \sqrt{3(2)+3}$ | |
| $= 4 - \sqrt{9}$ | |
| $= 4 - 3$ | |
| $= 1$ | |

$$\text{LS} \not< \text{RS}$$

So, $x = 2$ does not satisfy the original inequality.

The solution is $-1 \leq x < 2$.

The end values may also be included or excluded by checking the table of values.

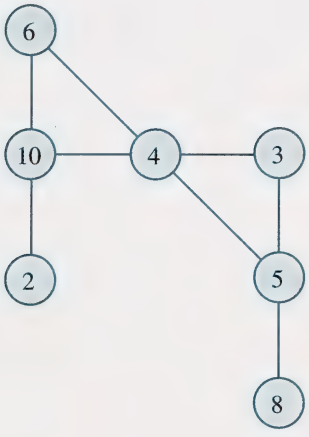



← $-1$ is included because $Y_1 = 1$.

← 2 is excluded because $Y_1 = 0$.

All $x$-values with a $y$-value of 1 are included.

**5. Textbook question "NUMBER POWER," p. 327**



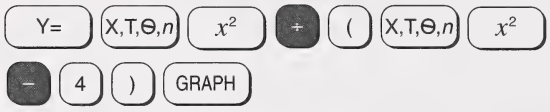
# Section 3: Follow-up Activities

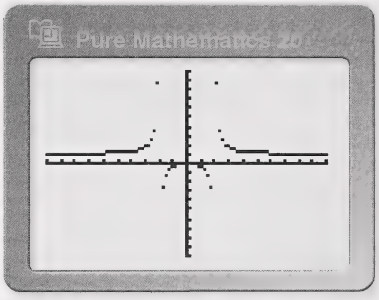
## Extra Help

**1. Textbook questions 1, 3, 5, and 9 of Investigation 1, "Exploring Rational Functions," p. 313**

**1.** $y = \dfrac{x^2}{x^2 - 4}$

Before graphing this function on your graphing calculator, for clarity, make sure the calculator is in Dot mode.




The horizontal asymptote is $y = 1$, and the vertical asymptotes are $x = -2$ and $x = 2$.

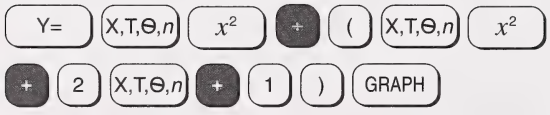
The domain is $x \neq \pm 2$, and the range is $y > 1$ or $y \leq 0$.

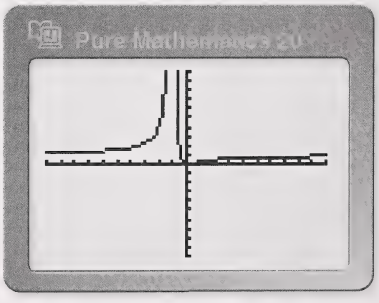
The real zero is 0.

The graph is symmetric with respect to the $y$-axis.

**3.** $y = \dfrac{x^2}{x^2 + 2x + 1}$

Use either Dot or Connected mode.




The horizontal asymptote is $y = 1$, and the vertical asymptote is $x = -1$.

The domain is $x \neq -1$, and the range is $y \geq 0$.

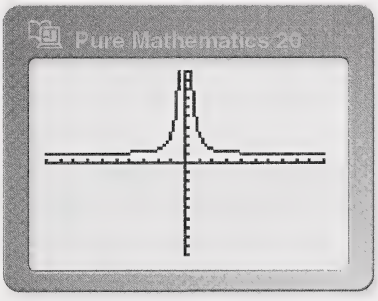
The real zero is 0.

There is no symmetry.

**5.** $y = \dfrac{x^2 + 2}{x^2}$

Use either Dot or Connected mode.

[Y=] [(] [X,T,Θ,n] [$x^2$] [+] [2] [)] [÷] [X,T,Θ,n] [$x^2$] [GRAPH]




The horizontal asymptote is $y = 1$, and the vertical asymptote is $x = 0$.

The domain is $x \neq 0$, and the range is $y > 1$.

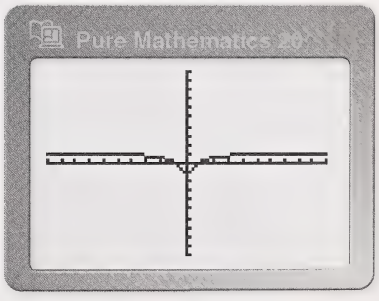
There are no real zeros.

The graph is symmetric about the $y$-axis.

**9.** $y = \dfrac{x^2 - 1}{x^2 + 1}$

Use either Dot or Connected mode.

[Y=] [(] [X,T,Θ,n] [$x^2$] [−] [1] [)] [÷] [(]
[X,T,Θ,n] [$x^2$] [+] [1] [)] [GRAPH]




The horizontal asymptote is $y = 1$, and there are no vertical asymptotes.

The domain is the set of reals, and the range is $-1 \leq y < 1$.

The real zeros are $-1$ and 1.

The graph is symmetric about the $y$-axis.

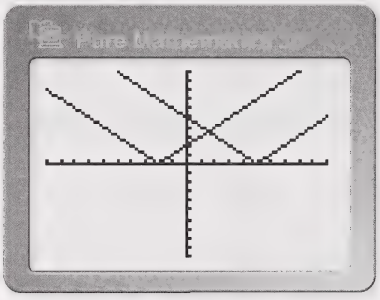
## Extra Help (continued)

**2. Textbook questions 48, 63, and 72 of "Review," pp. 330 and 331**

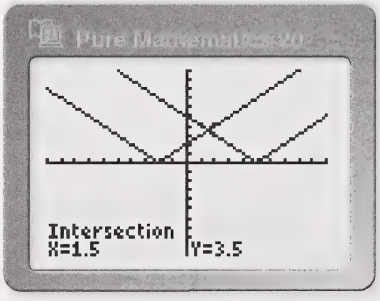
**48.** Solve $|x-5|<|x+2|$ using your graphing calculator.

Graph $y=|x-5|$ and $y=|x+2|$.

[Y=] [MATH] [ Select the NUM menu. ] [1] [X,T,Θ,n] [−] [5] [)] [ENTER] [MATH]
[ Select the NUM menu. ] [1] [X,T,Θ,n] [+] [2] [)] [GRAPH]



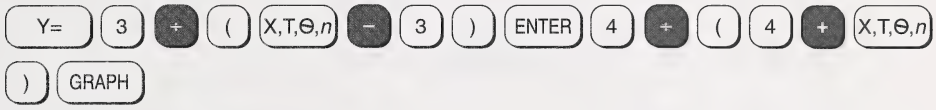
Next, determine where the graphs cross using the Intersect feature from the CALCULATE menu.

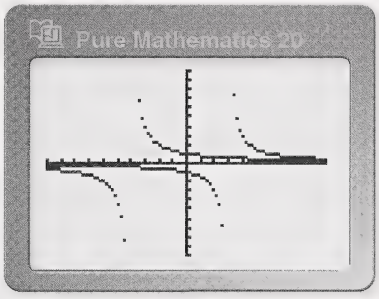



The graph of $y=|x-5|$ lies below the graph of $y=|x+2|$ to the right of the point of intersection.

The solution is $x>1.5$.

**63.** Solve $\frac{3}{x-3} < \frac{4}{4+x}$ using your graphing calculator. Make sure your calculator is in Dot mode. Graph $y = \frac{3}{x-3}$ and $y = \frac{4}{4+x}$.




The left-most branch of $y = \frac{3}{x-3}$ lies below $y = \frac{4}{4+x}$ between $x = -4$ and $x = 3$. Using the table and advancing it with the arrow keys shows an intersection at $x = 24$.

Check the interval $x > 24$. Try $x = 25$.

| LS | RS |
|---|---|
| $\frac{3}{25-3}$ | $\frac{4}{4+25}$ |
| $= \frac{3}{22}$ | $= \frac{4}{29}$ |
| $= 0.1364$ | $= 0.1379$ |

$$\text{LS} < \text{RS}$$

The solution is $-4 < x < 3$ and $x > 24$.

**Note:** Do not include $x = -4$ or $x = 3$ because $y = \frac{3}{x-3}$ is not defined for $x = 3$, and $y = \frac{4}{4+x}$ is not defined for $x = -4$.

## Extra Help (continued)

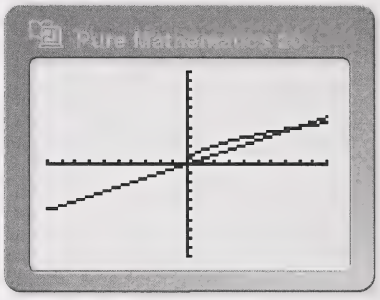
**72.** $\sqrt{2x} \le \frac{x}{2}$

Solve this inequality using your graphing calculator.

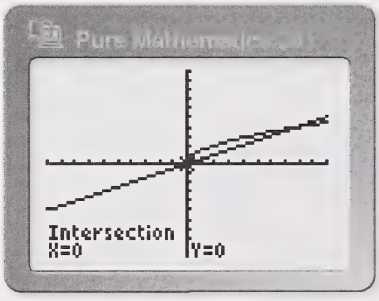
Graph $y = \sqrt{2x}$ and $y = \frac{x}{2}$.

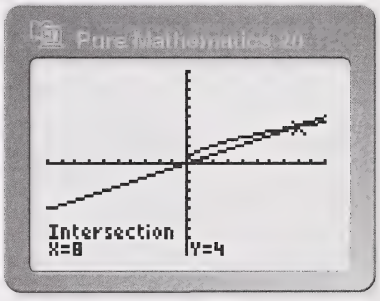
Y= 2nd [$\sqrt{\ }$] 2 X,T,Θ,*n* ) ENTER

X,T,Θ,*n* + 2 GRAPH



Determine the points of intersection using the Intersect feature from the CALCULATE menu.




The graphs intersect at $x = 0$ and at $x = 8$.

The graph of $y = \sqrt{2x}$ lies on or below the graph of $y = \frac{x}{2}$ when $x \ge 8$.

Because the graph of $y = \sqrt{2x}$ is not defined when $x < 0$, the solution is $x \ge 8$.

## Enrichment

**Textbook questions 1 to 11 of Investigation 1, "The Earth and the Moon," pp. 328 and 329**

**1.** $F = \dfrac{Gm_1m_2}{d^2}$

$= \dfrac{6.673\times10^{-11}\,(1)\left(5.97\times10^{24}\right)}{\left[6.37\times10^{6}\right]^2}$

$\doteq 9.8$

The weight of a 1-kg mass on Earth's surface is approximately 9.8 N.

**2.** $W = 9.8m$

**3.** $W = 9.8m$

$= 9.8(60)$

$= 588$

The weight of a 60-kg person on Earth's surface is approximately 588 N.

**4.** $F = \dfrac{Gm_1m_2}{d^2}$

$= \dfrac{\left(6.673\times10^{-11}\right)(1)\left(7.34\times10^{22}\right)}{\left(1.74\times10^{6}\right)^2}$

$1740\ \text{km} = 1.74\times10^{6}\ \text{m}$

$\doteq 1.6$

The weight of a 1-kg mass on the Moon's surface is approximately 1.6 N.

**5.** $W = 1.6m$

**6.** $W = 1.6m$

$= 1.6(60)$

$= 96$

The weight of a 60-kg person on the Moon's surface is approximately 96 N.

## Enrichment (continued)

**7.** Height of hurdles on the Moon $= \dfrac{9.8}{1.6} \times 0.84$

$\doteq 5.1$

The women's hurdles should be about 5.1 m high on the Moon.

**8. a.** $v = \sqrt{\dfrac{2Gm}{r}}$

$= \sqrt{\dfrac{2\left(6.673 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)}{6.37 \times 10^{6}}}$

$\doteq 11\,200$

The escape velocity is approximately 11 200 m/s.

**b.** Convert m/s to km/h.

$$11\,200 \text{ m/s} = \frac{11\,200 \text{ m}}{1 \text{ s}} \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ h}}$$

$$\doteq 40\,300 \text{ km/h}$$

**9. a.** $v = \sqrt{\dfrac{2Gm}{r}}$

$= \sqrt{\dfrac{2\left(6.673 \times 10^{-11}\right)\left(7.34 \times 10^{22}\right)}{1.74 \times 10^{6}}}$

$\doteq 2400$

The escape velocity from the Moon is approximately 2400 m/s.

**b.** Convert m/s to km/h.

$$2400 \text{ m/s} = \frac{2400 \text{ m}}{1 \text{ s}} \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ h}}$$

$$\doteq 8600 \text{ km/h}$$

**10.** If Earth's mass remained the same and its radius decreased, the weight of an object on Earth's surface would increase. The gravitational force between two objects is inversely proportional to the square of the distance between the centres of mass of the objects. For example, halving the distance quadruples the force.

**11.**

$$\frac{F}{F_{\text{surface}}} = 25\%$$

$$\frac{F}{F_{\text{surface}}} = 0.25$$

$$\frac{\frac{G m_{\text{spacecraft}} \times m_{\text{Earth}}}{d^2}}{\frac{G m_{\text{spacecraft}} \times m_{\text{Earth}}}{\left(6.37\times10^6\right)^2}} = 0.25$$

$$\frac{\frac{1}{d^2}}{\frac{1}{4.06\times10^{13}}} = 0.25$$

$$\frac{4.06\times10^{13}}{d^2} = 0.25$$

$$0.25d^2 = 4.06\times10^{13}$$

$$d^2 = \frac{4.06\times10^{13}}{0.25}$$

$$d \doteq 1.27\times10^7$$

The weight of a spacecraft would equal 25% of its weight on Earth when the spacecraft would be about $1.27\times10^7$ m from Earth's centre. This is $1.27\times10^7 - 6.37\times10^6 = 6.37\times10^6$ m above Earth's surface. This distance is about 6370 km.

Pure Mathematics 20
Student Module Booklet
Module 4

LRC

1999, 2001